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THESIS

FINITE ELEMENT GRID OPTIMIZATION

by

Jesse Blair Schrum

September 1983

Thesis Advisor: R. E. Newton

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Since widespread use of the finite element method began in the early 1960's, much effort has been devoted to the development of the method itself, while only recently has there been any research directed at minimizing the discretization error by a proper selection of the element grid. This paper examines some recently proposed grid optimization techniques and applies them to some one- and two-dimensional linear self-adjoint boundary value problems. Guidelines requiring minimal software modification are recommended to assist the analyst in obtaining improved finite element solutions.

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Finite Element Grid Optimization

by

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Lieutenant, United States Navy
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Submitted in partial fulfillment of the
requirements for the degrees of

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ABSTRACT

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TABLE OF CONTENTS

I. INTRODUCTION 9
A. HISTORICAL BACKGROUND 9
B. INVESTIGATIVE APPROACH 11
C. OBJECTIVES 13

II. CRITERIA FOR GRID REFINEMENT 14
A. SOLUTION PARAMETER VARIATION 14
B. GRID ITERATION 15
C. ELEMENT RESIDUALS 16
D. A-POSTERIORI ERROR ESTIMATES 17

III. METHODS OF GRID REFINEMENT 19
A. CONVERGENCE OF GRID REFINEMENT 20
B. HIERARCHICAL FINITE ELEMENTS 23

IV. GRID OPTIMIZATION TECHNIQUES 26
A. MATHEMATICAL PROGRAMMING 27
B. CONTOURING 29
C. SELECTIVE REFINEMENT 31
D. SUBDOMAIN ISOLATION 34
E. MESH GRADING 35

V. APPLICATION TO ONE-DIMENSIONAL PROBLEMS 38
A. ELASTIC CABLE PROBLEM 39
B. TAPERED BAR PROBLEM 44
C. GUIDELINES FOR ONE-DIMENSIONAL GRID OPTIMIZATION 50

VI. APPLICATION TO TWO-DIMENSIONAL PROBLEMS 53
A. PROBLEM DESCRIPTION 54
B. COMPUTER IMPLEMENTATION 56

C. ASYMPTOTIC ERROR ANALYSIS FOR UNIFORM REFINEMENT 57
D. PROBLEM SOLUTION WITH GRID OPTIMIZATION . . . 63
1. Contouring 63
2. Selective Refinement 70
3. Subdomain Isolation 79
4. Mesh Grading 84
E. GUIDELINES FOR TWO-DIMENSIONAL GRID OPTIMIZATION 89

VII. CONCLUSION AND RECOMMENDATIONS 94

APPENDIX A: FORMULATION OF THE ELASTIC CABLE PROBLEM . . 96
A. PROBLEM STATEMENT 96
B. PROBLEM SOLUTION 97

APPENDIX B: FORMULATION OF THE TAPERED BAR PROBLEM . . . 98
A. PROBLEM STATEMENT 98
B. PROBLEM SOLUTION 99

APPENDIX C: FORMULATION OF THE TORSION PROBLEM 100
A. PROBLEM STATEMENT 100
B. PROBLEM SOLUTION 101

LIST OF REFERENCES 103

INITIAL DISTRIBUTION LIST 106

LIST OF TABLES

I. Tension Cable Problem Solution Results 42
II. Tapered Bar Problem Solution Results 47
III. Reversed Tapered Bar Solution Results 50
IV. Contouring Solution Results 69
V. Selective Refinement Solution Results 75
VI. Selective p-Version Solution Results 78
VII. Subdomain Isolation Solution Results 83
VIII. Mesh Grading Solution Results 88
IX. Comparison of Two-Dimensional Results 90

LIST OF FIGURES

4.1	Some h-Version Subdivision Schemes	32
4.2	Graded Mesh for a Perforated Square Plate	37
5.1	Tension Cable Deflection on Elastic Foundation	40
5.2	Tension Cable Refinements	41
5.3	Linearly Tapered Bar Under Axial Loading	44
5.4	Tapered Bar Refinements	46
5.5	Reversed Tapered Bar with Distributed Load	48
5.6	Reversed Tapered Bar Refinements	49
6.1	Cross-section of Shaft with Keyway	54
6.2	Uniform Linear and Quadratic Element Grids	58
6.3	Keynode Placement for Isoparametric Mapping	59
6.4	Asymptotic Convergence for Uniform Refinement	61
6.5	Stress Distribution on Shaft Keyway	62
6.6	Contouring for the Torsion Function	65
6.7	Contouring for the Shear Stress Function	66
6.8	Contouring for the SED Function	67
6.9	Selective Refinement Procedure - Linear Elements	72
6.10	Selective Refinement Procedure - Quadratic Elements	73
6.11	Selective Refinement Employing p-Version	77
6.12	Subdomain Isolation - Linear Elements	80
6.13	Subdomain Isolation - Quadratic Elements	81
6.14	Mesh Grading - Linear Elements	86
6.15	Mesh Grading - Quadratic Elements	87
A.1	Tension Cable Under Distributed Loading	96

B.1 Tapered Bar with Applied Loads 98
C.1 Cross-section of Shaft with Keyway 100

I. INTRODUCTION

The critical concern in any finite element analysis is the adequacy of the selected mesh to provide reliable solution results within some reasonable cost range. The goal of finite element grid optimization then becomes one of obtaining maximum solution accuracy for a prescribed analysis cost. While this objective is generally not realized in today's widespread use of finite element analysis, the efficient computation of solutions with optimal accuracy will become paramount to the engineer as finite element methods are applied to increasingly difficult dynamic, nonlinear, and evolution problems.

A. HISTORICAL BACKGROUND

In the early 1960's, with the help of the high speed digital computer, finite element methods revolutionized problem solving in engineering. Since that time the major research efforts have concentrated on expanding the theoretical basis of the method and extending its application in a variety of fields. Only recently has there been significant attention directed at minimizing finite element solution errors by properly defining the element grid. Early attempts at distributing the nodes and choosing the elements to ensure some degree of confidence in the solution accuracy were predominantly dependent upon the analyst's engineering judgement and experience, since there were no established procedures for accomplishing this objective. Even these attempts towards grid optimization have become largely ignored with the advent of automatic mesh generators, which have drastically reduced preprocessing costs while

accomplishing little in improving solution accuracy. These programs automatically construct the element grid, usually in a uniform manner, after the user merely defines the problem and specifies the number of elements to be used. To establish convergence and achieve the desired solution accuracy, the user simply repeats the analysis using a finer mesh of uniformly distributed elements while monitoring such convergence indicators as successive solution values at common nodal points or the asymptotic increase in the energy content of the mesh. The often disastrous flaw in such a practice is that for nearly degenerate problems which exhibit very large gradients, the asymptotic range is only entered for an extremely large number of degrees-of-freedom, often beyond computer limitations [Ref. 1]. In this case, uniform mesh refinement may produce apparent convergence, when in fact the solution accuracy is poor. Therefore, the need for a finite element grid optimization procedure is clearly evident.

The first formal attempts at finite element grid optimization did not begin until the early 1970's. These early approaches involved the inclusion of the nodal coordinates as dependent variables in the minimization of the potential energy functional [Ref. 2]. Unfortunately, the resulting system of equations is highly nonlinear and the computational effort involved in its solution is so great that similar accuracy can be obtained at a fraction of the expense, simply by employing a very fine mesh. Clearly, this method does not approach the finite element grid optimization goal of achieving a specified solution accuracy for a minimum analysis cost. For this reason, virtually all of the grid optimization techniques since developed are based on a "near-optimum" strategy whereby nearly-optimal solution results are obtained without the computational inefficiency of a numerical optimization analysis. The growing emphasis

has been on adaptivity, a procedure for efficient construction of near-optimum grids by the iterative application of some criterion, based on data already computed from the solution for a previous grid. This procedure is far more efficient than the conventional approach of repeating the analysis using successively finer uniform grids. Experimental self-adaptive finite element codes have recently been developed. Starting from the user's initial idealization, these programs automatically generate a near-optimum grid and solve the resulting equations.

B. INVESTIGATIVE APPROACH

In undertaking any numerical optimization task, the analyst must first define the objective along with any constraints to be imposed upon the objective variables; and finally a numerical algorithm must be prescribed to perform the optimization, preferably one which will do so efficiently for the particular problem. Since the term "optimum" most often refers to a solution obtained by mathematical programming, which is very inefficient for grid optimization, a near-optimum grid obtained by application of an adaptive procedure, henceforth will be termed an optimum grid. However, before such a grid can be determined to satisfy the stated objective of obtaining maximum solution accuracy for a prescribed cost, the terms "accuracy" and "cost" must be defined; but, more importantly, the optimization goals must be specified. This is critical because grid optimization can be implemented in various forms depending upon the optimization goals, which will, in general, be determined by the original purpose for performing the finite element analysis [Ref. 3]. For example, if the purpose of the analysis is to evaluate a local quantity, such as the maximum value of the dependent variable or one of its

derivatives, then the nodal distribution should be densest in the region where that maximum is determined. If, on the other hand, the interest is on an integral quantity of the dependent variable over a region of the domain, then the nodes should be assigned to achieve a uniform distribution of the error over that region. For the purpose of this investigation, attention will be focused on the three finite element resultants with the most significance in solid mechanics and other fields in which finite element analysis is employed: the maximum value of the dependent variable, or solution; the maximum value of the gradient of the solution; and the integral of the solution over the domain.

In order to define the solution accuracy, it will be necessary to compare the error in the solution resultant obtained using an optimal grid to the error obtained using some baseline grid with the same number of degrees-of-freedom. For convenience, the reference grid chosen will be a uniform grid, or one with all elements of the same order and approximately the same size, with the understanding that no knowledgeable analyst would attempt to use such a grid in the solution of a problem with large gradients.

The definition of analysis cost will be greatly simplified by making the assumption that it is directly proportional to the number of degrees-of-freedom used to obtain the solution. In reality the cost depends on many factors, some of which are very difficult to quantify. Understandably, the number of degrees-of-freedom is not the sole measure of computational costs, but it appears to be an adequate measure of preprocessing and postprocessing costs which often account for the major portion of the total analysis.

This investigation will concentrate on the use of finite element grid optimization methods for solving problems of structural mechanics. While this area has dominated the

literature on the subject, the techniques presented herein extend equally as well to any field for which variational principles apply.

There are two key questions which must be answered prior to the adaptive application of finite element grid optimization:

(1) What criterion, based on the results of an initial finite element analysis, should be used to indicate regions where the original grid is inadequate ?

(2) What method of grid refinement should be employed ?

Considerable attention will be devoted to these questions in the next two chapters.

C. OBJECTIVES

The objectives of this investigation are:

(1) To examine some recently developed grid optimization techniques which have reached the state of practical application.

(2) To implement some of these techniques in the solution of some one- and two-dimensional linear self-adjoint boundary-value problems.

(3) To draw a comparison among these applied techniques in terms of solution accuracy, analysis cost, and ease of implementation.

(4) To recommend some guidelines to assist the analyst in obtaining optimal finite element solutions employing currently available or easily amendable software.

II. CRITERIA FOR GRID REFINEMENT

The primary thecretical concern in the application of adaptive grid optimization is the selection of the refinement criterion. In cther words, one must decide upon which solution parameter, obtained from an initial idealization, may most appropriately be used to give some indication as to where the initial grid is inadequate and thus needs refinement. There are several competing proposals concerning the most appropriate choice of a refinement criterion. In reality, the decision must be based upon such factors as the type of problem being solved, which criterion is most practically implemented, and whether accuracy is desired locally, globally, pointwise, or with respect to an integral quantity. The following are some of the more practical refinement criteria used in grid optimization at present.

A. SOLUTION PARAMETER VARIATION

The most direct, computationally inexpensive, and earliest proposed indication of where an element grid requires refinement is a measure of the variation of some solution parameter over the domain. It is only logical that a piecewise polynomial approximation would experience the most difficulty in modeling the desired response in those regions where the solution or its resultants were either not smooth or were characterized by large gradients. Therefore, the basis of this criterion is to refine the mesh in those areas where a solution parameter varies rapidly, with the implication that the optimum mesh is one for which the solution parameter variation over each element is uniform throughout the domain. The first consideration in the

application of such a criterion is to find a scheme for distributing the nodes to achieve such a condition. For one-dimensional problems the task is trivial, but one way to ensure equal variation over each element in higher dimensions is to position the nodes along equidistant contours of the chosen parameter. This is precisely the procedure prescribed for a practical optimization technique known as contouring. The other consideration is the determination of which solution parameter is to be used. In fact, several solution parameters have been found to work quite well [Ref. 4], but the most commonly used and the one that is consistent with the potential energy minimization formulation is the strain energy density [Ref. 2]. Because its employment requires only minor software changes and it has been found to produce excellent results, this refinement criterion was used extensively throughout the course of this research.

B. GRID ITERATION

Another, rather basic but less direct, method of locating regions of the mesh to be refined is known as grid iteration, which can be implemented in one of two ways. An initial coarse grid analysis may be repeated using either a finer or a higher order mesh, and a comparison of the resultants of interest between the two solutions will identify those areas of the domain where refinement is most effective. Another approach is based on the assumption that the greatest benefit is to be gained by refinement in those regions where a small perturbation, like the introduction of a single additional degree-of-freedom, produces the greatest change in the solution or one of its resultant parameters. Since additional degrees-of-freedom would be expected to produce the greatest change in those regions where the

desired response varied most rapidly, refinements based on this method provide results very similar to those obtained using the solution parameter variation criterion already discussed. The grid iteration method may at first appear to be more computationally expensive, but it was developed to be most efficiently implemented employing a special family of elements. These elements, called "hierarchical", possess some very desirable properties for this application and will be discussed in the next chapter.

C. ELEMENT RESIDUALS

The major drawback with refinement criteria based on solution parameter variations is that their applicability appears limited to elastostatic problems. For this reason, several investigators have recently developed grid refinement criteria based on element residuals, which appear promising for application to all types of finite element problems, including nonlinear analysis. The reason for this is that the residual has essentially the same meaning regardless of the problem type [Ref. 5]. For example, consider the governing differential equation,

$$D[\,u\,] - f = 0$$

defined on some domain, where $D[\ ]$ is a linear or nonlinear differential operator, and the dependent variable u and the non-homogeneous term f are both functions of the independent variables. Let the finite element approximation to the solution of the differential equation be $\tilde{u} \cong u$. Applying the differential operator to the approximation gives rise to the residual, which is defined as

$$R = D[\,\tilde{u}\,] - f$$

and is not identically zero unless the finite element solution is exact.

The key to using the residual as a criterion for determining regions of the domain where refinement is necessary is the local residual on the element level, which indicates the contribution of the element to the total error of the finite element approximation. Since the residual is a pointwise quantity, the useful measure of the element error contribution is a norm of the element residual, or the integral over the element of the product of the residual and some weight function. The integration is most readily performed using numerical quadrature. The grid optimization strategy then becomes one of refining the mesh so as to equi-distribute the element residual norms, by forcing them to become smaller and more uniform over the domain.

There are some drawbacks to the element residual refinement criterion which have not yet been fully resolved, such as appropriate selections of the residual norm and the weight function, and in the computation of the residual itself. While the evaluation of the residual presents no particular difficulties in the interior of the element, it is rarely determinable at the edges. The reason for this is that in formulating the finite element approximation the element shape functions are defined so as to provide only that degree of continuity required to adequately model the physical problem; the most frequent consequence being that $D[\tilde{u}]$ is undefined along the interelement boundaries. Unfortunately, this singularity cannot be ignored and a more complicated analysis must be applied in order to bound the residual contributions at these boundaries [Ref. 6].

D. A-POSTERIORI ERROR ESTIMATES

A sophisticated extension of the element residual criterion is one based on computable error estimates from an initial finite element analysis. This utilizes the energy

norm of the residual, in which case the weight function is the residual itself. Research in reliable error estimates was pioneered by Babuska [Ref. 7, 8] for linear quadrilateral elements and more recently by Zienkiewicz [Ref. 9] for higher order elements. The major difference from the residual criterion is that instead of equi-distributing the element residual norms over the domain, they are normalized to compute error indicators for the elements, which are in turn used to compute reliable pointwise error estimates for the solution as well as the energy error over the domain. These quantities are of primary importance because they provide not only an indication of where additional refinement is most effective, but also a measure of the quality of the mesh to determine whether additional refinement is necessary [Ref. 9]. The optimization strategy is to obtain a nearly uniform distribution of the error indicators throughout the domain, which corresponds to minimizing the error in the energy norm. The refinement procedure may progress until all the local errors are within some prespecified tolerance. While the practical utility of such a refinement criterion is obvious, the mathematical development and the algorithms involved are rather complicated. However, the process is not computationally expensive, and there now exists a prototype self-adaptive finite element code, FEARS, which implements this refinement criterion [Ref. 10].

III. METHODS OF GRID REFINEMENT

Once it has been determined where the initial element grid is inadequate and needs refinement, the next consideration is how the idealization in these areas should be improved. The choice of the refinement method to be employed may well be a more important decision than the selection of one of the refinement criteria previously discussed, since at least one investigator has observed that for a particular method of grid refinement, the various refinement criteria produce essentially the same solution results [Ref. 11].

Grid refinement is the process of introducing additional degrees-of-freedom into selected regions of the finite element grid, and may be performed by one of three methods:

(1) The polynomial degree of the elements remains fixed, usually at a low value, while the size of the elements is reduced. This has become known as the h-version since element size is commonly denoted by the letter h.

(2) The size of the elements, usually few in number, remains fixed while the polynomial degree of the elements is increased. This has become known as the p-version since polynomial order is commonly denoted by the letter p.

(3) The size of the elements may be reduced concurrently with an increase in their polynomial order. This is known as the combined h- and p-version of the finite element method.

A. CONVERGENCE OF GRID REFINEMENT

It is well known that the finite element method converges with an increasing number of degrees-of-freedom; in fact, this is the justification for its development. Therefore, the appropriate measure of the effectiveness of a particular grid refinement method should be the associated rate of convergence, which generally will be affected by the smoothness of the approximated function over the subdomain of interest. It has been demonstrated that when the refinement is performed uniformly over the domain, the associated rate of convergence is asymptotic, provided the number of degrees-of-freedom is sufficiently large [Ref. 1]. The asymptotic rate of convergence is often measured as the slope of the error versus cost curve in the linear, or asymptotic, range when plotted on a log-log scale. In performing such an error analysis for the displacement formulation of the finite element method, the error is usually the relative strain energy error, approximated by the energy norm, and the cost is assumed to be some simple function of the average element size or the number of degrees-of-freedom [Ref. 12: p. 726]. Only in the past several years has there been any significant research comparing the relative merits of the different methods of grid refinement. Since the solutions of elliptic boundary value problems are usually very smooth over convex domains except in the vicinity of corners, most of this research has focused on solutions exhibiting singularities, which severely hinder the rate of convergence, as in problems of fracture mechanics and in domains with re-entrant corners [Ref. 1, 13, 14, 15].

In order for a finite element analysis to be both efficient and reliable, the asymptotic convergence range should be entered for as few degrees-of-freedom as reasonably

possible. In general the p-version satisfies this requirement better than the h-version. While it has been established that p-convergence will necessarily occur whenever h-convergence occurs, the converse is not true. For example, if the h-version using a uniform grid of linear elements is applied to a nearly degenerate problem, the number of degrees-of-freedom required for entry into the asymptotic range may be beyond the computer's round-off limitations, in which case convergence will not occur unless the polynomial order is increased [Ref. 1]. Numerical experiments on such problems clearly indicate that the p-version requires considerably fewer degrees-of-freedom than the h-version to achieve the same degree of accuracy. Recent analyses [Ref. 1, 13] of the asymptotic rate of convergence in energy for non-smooth solutions, using uniform refinement with sufficiently high numbers of degrees-of-freedom, have demonstrated that the p-version cannot have a slower rate of convergence than the h-version. Furthermore, if the singularity is confined to element boundaries, as is usually the case, the error for p-method is inversely proportional to the number of degrees-of-freedom, whereas the error is inversely proportional to the square root of the number of degrees-of-freedom in the h-version. In other words, for this special class of problem, the rate of convergence for the p-version is twice that for the h-version, which is due primarily to the ability of higher order polynomials to "absorb" singularities occurring at the element boundaries. This implies, at least for this type of problem, that in order to minimize the error for a specified number of degrees-of-freedom, the best strategy is not to subdivide the domain uniformly, but to use instead a single element of increasing polynomial order [Ref. 15].

Since it is unlikely that one would attempt to solve such a problem using uniformly finer grids, a more useful comparison between the convergence rates of the two versions would be based on adaptive refinement employing one of the solution-based criteria discussed in the previous chapter. It so happens that the h-version, when used with optimally refined meshes, can have a higher convergence rate than the uniformly distributed p-version, provided that the element order is sufficiently high. However, the p-version can also be employed with an optimal refinement criterion. While there are yet no proven theorems concerning the convergence rates for non-uniform refinement, obtaining the desired solution accuracy with optimal p-distributions appears to be much less sensitive to the particular choice of the elements to be refined than with optimal h-refinement [Ref. 13].

It would seem plausible that an even better optimization strategy would involve a proper combination of both the h- and p-versions. It has been demonstrated for problems with corner singularities, that a proper sequence of h-refinements combined concurrently with the proper sequence of p-distributions results in extremely high convergence rates, conjectured to be exponential [Ref. 15]. However, this proper combination is difficult to determine, and adaptive refinement based on the combined h- and p-versions poses some difficult data management problems. To avoid this problem a more promising approach, proposed by Babuska and Szabo [Ref. 1], employs a graded mesh in which the element sizes are first reduced according to a geometric progression towards the singularity, followed by determining the optimal p-distribution for those elements using an adaptive criterion. However, obtaining the optimal combination when employing this scheme can be a delicate matter and, astoundingly, the highest accuracy is achieved when the polynomial order of the elements actually increases with distance from the singularity.

There are some additional advantages of the p-version worth mentioning. Because the p-version employs fewer elements, there are lesser preprocessing and postprocessing costs than for the h-version. Furthermore, when bandwidth minimization and sparse matrix solution techniques are used, the solution time for the p-version is approximately the same as for the h-version for a specified number of degrees-of-freedom, and the p-version appears less susceptible to round-off errors. Finally, the p-version is simpler to implement adaptively than the h-version when hierarchical elements are employed [Ref. 13].

B. HIERARCHICAL FINITE ELEMENTS

The hierarchical concept was first introduced as a simple method for implementing the p-version and as a convenient device for imposing boundary continuity between elements of different polynomial order [Ref. 9]. Since then a useful family of elements based on the hierarchical concept has been developed and incorporated into COMET-X, an experimental finite element code, developed by Szabo, which self-adaptively employs both the h- and p-versions of the finite element method [Ref. 14].

For a brief description of the hierarchical concept consider the conventional finite element formulation which produces the following system of equations:

$$\underset{\sim}{K}^{(n)} \underset{\sim}{u}^{(n)} = \underset{\sim}{f}^{(n)} \qquad \text{(Eqn. 3.1)}$$

where n is the number of degrees-of-freedom, $\underset{\sim}{K}^{(n)}$ is the $n \times n$ global stiffness matrix, $\underset{\sim}{u}^{(n)}$ is the finite element approximation of the exact solution, and $\underset{\sim}{f}^{(n)}$ is the n-component global load vector. When m higher order

degrees-of-freedom are added to the original system using conventional refinement methods the system of equations becomes:

$$\underset{\sim}{K}^{(n+m)}\,\underset{\sim}{u}^{(n+m)} = \underset{\sim}{f}^{(n+m)} \qquad \text{(Eqn. 3.2)}$$

where the contributions to $\underset{\sim}{K}^{(n+m)}$ and $\underset{\sim}{f}^{(n+m)}$ from the refined elements result in a completely different set of coefficients. If, on the other hand, this refinement had been made hierarchically, the equations would become:

$$\begin{bmatrix} \underset{\sim}{K}^{(n)} & \underset{\sim}{K}^{(n,m)} \\ \underset{\sim}{K}^{(m,n)} & \underset{\sim}{K}^{(m,m)} \end{bmatrix} \begin{bmatrix} \underset{\sim}{u}^{(n)} \\ \underset{\sim}{u}^{(m)} \end{bmatrix} = \begin{bmatrix} \underset{\sim}{f}^{(n)} \\ \underset{\sim}{f}^{(m)} \end{bmatrix} \qquad \text{(Eqn. 3.3)}$$

where $\underset{\sim}{K}^{(n)}$ and $\underset{\sim}{f}^{(n)}$ are the stiffness matrix and force vector from the original system of equations for n degrees-of-freedom appearing in Equation 3.1. However, $\underset{\sim}{u}^{(m)}$ is not the nodal values of the finite element solution for the m additional degrees-of-freedom, but instead represents the difference between those values and the pointwise values obtained from the lower order polynomial interpolation for the unrefined mesh of n degrees-of-freedom.

The primary advantage of hierarchical elements is immediately observable from Equation 3.3. Because the shape functions of an element of order p constitute a subset of the shape functions of an element of order p+1, the local stiffness matrix and force vector for each hierarchical element is embedded in the stiffness matrices and force vectors of all hierarchical elements of higher order. Therefore, the global stiffness matrix $\underset{\sim}{K}^{(n)}$ and force vector $\underset{\sim}{f}^{(n)}$ of the original system are preserved, thus saving

considerable time and effort expended on computing the coefficients for successive refinements [Ref. 14]. Another advantage is that the hierarchical form of the global stiffness matrix is more diagonally dominant than the one resulting from a conventional refinement, resulting in improved conditioning and faster convergence when iterative solvers are employed [Ref. 9]. Another benefit of hierarchical elements, which arises from the "add-on" nature of the nodal variables of the higher order degrees-of-freedom, is that the problem of maintaining boundary continuity between elements of different polynomial order becomes trivial. Instead of introducing global constraint equations for the higher order degrees-of-freedom, the nodal variables are simply set equal to zero and condensed out, as if they were zero-valued Dirichlet boundary conditions [Ref. 2].

There are two major drawbacks with hierarchical elements that have hampered their widespread acceptability. The first, which has already been mentioned, is that the nodal variables for the higher order degrees-of-freedom represent difference values rather than the more easily identifiable values of the dependent variable itself. Secondly, when implementing the h-version of the finite element method, special integration rules must be introduced when the subdivided element is in hierarchical form [Ref. 9]. Of course, the latter problem can be evaded by using the p-version, for which the hierarchical concept was developed. In spite of the disadvantages of hierarchical elements, their considerable computational efficiency and utility for grid optimization will certainly result in their widespread utilization in future adaptive finite element codes.

IV. GRID OPTIMIZATION TECHNIQUES

Once the analyst has identified where the initial grid needs enrichment and decided which refinement method to employ, he must then determine a systematic procedure, or algorithm, to perform the refinement according to the criterion selected. The ultimate goal of such a procedure is to design an element grid which meets the optimization objective of obtaining maximum solution accuracy for a specified analysis cost. While the analyst may or may not have an indication of the accuracy of the solution, he should have a preconceived notion of cost, or how much effort he is willing to expend to arrive at a better solution. Therefore, with some knowledge of the grid optimization techniques available and an understanding of the advantages and disadvantages of each, the analyst can realize the grid optimization goal.

There are essentially two adaptive grid optimization strategies:

(1) Grid refinement, in which the initial analysis is performed on a relatively coarse grid, and new degrees-of-freedom are added to the same grid by the iterative application of the solution-based criterion.

(2) Grid modification, in which the initial analysis is performed using a prespecified number of degrees-of-freedom, and the solution-based criterion is employed to shift degrees-of-freedom from certain regions to others. This may involve complete grid redefinition in an effort to obtain a near-optimum grid in a single cycle.

Much of the interest lately has been in the development of complicated self-adaptive software packages which minimize the impact of the user's skill on the final solution. Ideally, the analyst would merely define the problem and the program would automatically generate and analyze the optimum grid employing one or more of these techniques, possibly at the user's option.

A. MATHEMATICAL PROGRAMMING

No discussion of grid optimization techniques would be complete without a brief description of mathematical programming, not only because it is how grid optimization was earliest attempted, but more importantly, it is precisely what the engineer envisions when he hears the term "optimization". It is not a grid optimization technique, per se, but rather a numerical process of achieving any optimization objective, once it is explicitly defined in mathematical terms. In solid mechanics the finite element method is a numerical method for minimizing the potential energy functional, which in discretized form may be written:

$$\Pi = \tfrac{1}{2}\, \underset{\sim}{u}^T \underset{\approx}{K}\, \underset{\sim}{u} - \underset{\sim}{u}^T \underset{\sim}{f} \qquad \text{(Eqn. 4.1)}$$

where: $\underset{\sim}{u}$ is the global displacement vector

$\underset{\approx}{K}$ is the global stiffness matrix, and

$\underset{\sim}{f}$ is the global load vector

In the classical finite element formulation, the potential energy is minimized with respect to the nodal displacements, which implies satisfaction of the following stationary conditions:

$$\frac{\partial \pi}{\partial u_i} = 0 \qquad (i = 1,2,\ldots,n) \qquad \text{(Eqn. 4.2)}$$

where n is the number of degrees-of-freedom. This leads to the very familiar system of linear equations:

$$\underset{\approx}{K}\,\underset{\sim}{u} - \underset{\sim}{f} = \underset{\sim}{0} \qquad \text{(Eqn. 4.3)}$$

However, since $\underset{\approx}{K}$ and $\underset{\sim}{f}$ are functions of the nodal coordinates, then the potential energy could be minimized with respect to the nodal coordinates as well. This would require satisfaction of the following additional stationary conditions:

$$\frac{\partial \pi}{\partial x_j} = 0 \qquad (j = 1,2,\ldots,m) \qquad \text{(Eqn. 4.4)}$$

where m is the number of nodal coordinates, x_j. This differentiation leads to the less familiar system of non-linear equations:

$$\tfrac{1}{2}\,\underset{\sim}{u}^T \frac{\partial \underset{\approx}{K}}{\partial x_j}\,\underset{\sim}{u} - \frac{\partial \underset{\sim}{f}^T}{\partial x_j}\,\underset{\sim}{u} = 0 \qquad (j = 1,2,\ldots,m) \qquad \text{(Eqn. 4.5)}$$

This, then, is the mathematical statement of the grid optimization problem for the elastostatic case. The nodal displacement variables may be eliminated by minimizing the potential energy with respect to the nodal coordinates only, subject to the implicit constraint that Equation 4.3 is always satisfied [Ref. 4]. Unfortunately this does not help much because the objective function is still nonlinear, rendering most numerical optimization algorithms inefficient and unreliable. The difficulty is even further compounded by

the requirement that the nodal variables be subject to side constraints in order to maintain the defined boundary of the domain and to ensure that the elements neither distort excessively nor overlap one another. For all except the simplest of problems, these constraints may be even more severely nonlinear than the objective function, resulting in the analysis becoming prohibitively expensive [Ref. 2]. For this reason, mathematical programming in finite element grid optimization has been abandoned in favor of some equally reliable, yet far more computationally efficient grid optimization techniques. However, these early efforts with mathematical programming were not totally in vain because they gave rise to the contouring techniques.

B. CONTOURING

Since mathematical programming is infeasible for grid optimization, further investigations were conducted to suggest some guidelines to enable the analyst to construct a grid with similar topological features of the numerically optimized grid without the computational effort involved. Turcke [Ref. 4], in employing mathematical programming in the solution of some simple two-dimensional elastostatic problems, observed that there was a very definite element pattern common among problems involving high strain gradients and that the nodes of the numerically optimized grid generally tended to be aligned along contours of some response function being modeled. Consequently, in performing analyses on grids whose construction was based on contours derived from an initial analysis, it was determined that the following provided grid characteristics in regions of high strain gradients similar to the numerically optimized grid, but at a fraction of the computational expense:

- contours of displacement
- contours of maximum principal stress
- contours of maximum shear stress
- contours of strain energy density (isoenergetics)
- principal stress trajectories (isostatics)

Since the strain energy density is the response which is consistent with the principle of minimum potential energy, isoenergetics are the most commonly used contours along which element edges are aligned [Ref. 4]. However, there still remains the question of how to position the nodes along the contours. For this reason, isostatics have become increasingly popular because the principal stress trajectories form a "flow net" of orthogonal curves which can guide the analyst in the layout of the elements [Ref. 16].

Since contouring involves the redefinition of the grid, as opposed to a grid refinement, its primary advantage is that the enriched mesh is not constrained to the element configuration of the previous mesh. Therefore, there is no limit to the amount of enrichment per cycle which can be performed and it is conceivable that an optimum mesh could be generated in a single cycle [Ref. 2]. However, while the computational cost of repeated analyses is reduced, the preprocessing costs involved in constructing the contours and redefining the mesh can become quite high, especially if the contours are complex. Algorithms for performing these tasks in two-dimensional domains have been proposed [Ref. 4, 11], but they are not extendable to three-dimensional problems. The major obstacle for two- and three-dimensional domains is that it is often difficult to constrain the element edges to the contours without the elements becoming elongated or distorted to the degree that numerical inaccuracies result. Another difficulty, not addressed in the literature, is how the contour increments

should be selected when the response function is non-monotonic over the domain.

C. SELECTIVE REFINEMENT

The most commonly employed grid optimization technique is that of selective refinement. As its name implies, selected elements from a given mesh are enriched while the original element grid remains essentially intact. The elements selected for refinement are determined by the iterative application of the solution-based criterion to indicate which elements contribute most to the solution error. The refinement can be performed by either the h-version or the p-version, or even the combined version if so desired, but the choice is most often predetermined by the capabilities of the available preprocessor. Since the addition of new degrees-of-freedom over several iterations can quickly enlarge the problem, it is advisable to perform the initial analysis with a reasonably coarse grid of optimally shaped elements, that is nearly square quadrilaterals or nearly equilateral triangles. This is especially important in the h-version where it is desirable to prevent the successive subdivision of elements from producing elongated new elements. One refinement technique which will ensure this is the so called "father-to-four sons" subdivision scheme in which a single quadrilateral or triangular element is replaced by four new ones by adding and connecting midside nodes on the edges of the original element as shown in Figure 4.1. The major difficulty in selective refinement arises when the addition of a node along an edge of the element to be subdivided creates a higher polynomial ordered edge for an adjacent element which is not to be subdivided. There results an incompatibility in the interpolation of the dependent variable along this interelement boundary. Such is

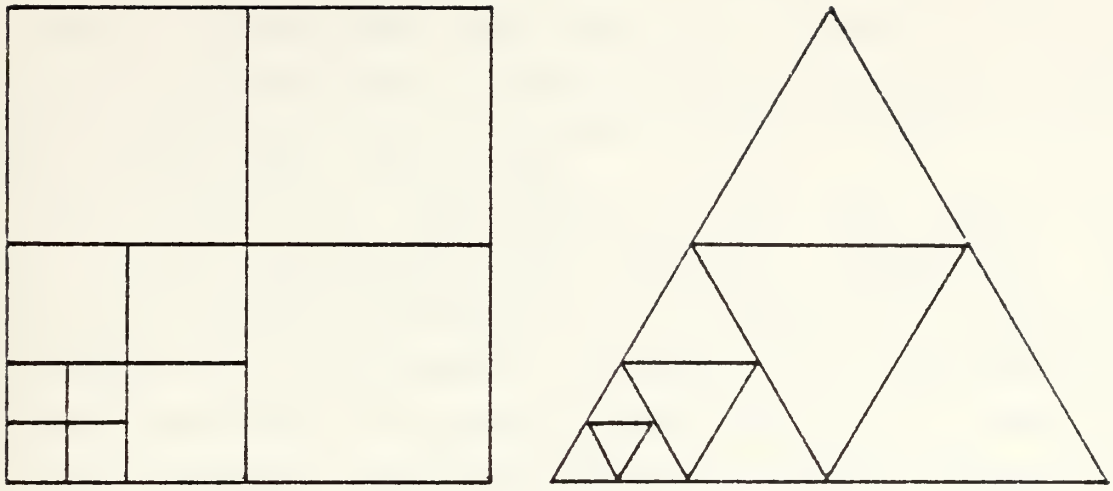


Figure 4.1 Some h-Version Subdivision Schemes.

the case in the h-version scheme of Figure 4.1 and it also arises in the p-version when two elements of different polynomial order share a common edge. When this situation occurs, the additional degrees-of-freedom do not actually represent degrees-of-freedom at all because they must be numerically constrained to the polynomial interpolant of the lower order. Such constraints are usually imposed in one of three ways: global constraint equations may be written; the constraints may be incorporated in the elemental basis; or hierarchical forms may be used with the excess degrees-of-freedom simply set to zero and condensed out [Ref. 2].

There are some other selective refinement techniques which do not require any major software modifications. In the h-version, the continuity problem may be circumvented by employing any of the coarse-to-fine mesh transition schemes for which all of the element edges remain of the same polynomial order [Ref. 17: p. 210]. However, it is impossible to employ these schemes without permitting some element

distortion, and the refinement must nearly always be performed interactively rather than automatically. For the p-version, interelement continuity can be easily ensured by employing variable-noded isoparametric elements, which permit a single element to possess edges of differing polynomial orders [Ref. 17: p. 125].

The analyst must also exercise care when adding new nodes to the boundary of the domain to ensure that the appropriate boundary conditions are determined and applied. Furthermore, if the boundary is curved, the coordinates of the new node should be computed such that it is placed on the actual boundary and not necessarily on the edge of the element being refined [Ref. 2].

The important advantage of the selective refinement technique is that once an appropriate refinement criterion has been determined, selecting candidate elements for refinement in each cycle becomes straightforward. The refinement can then be continued indefinitely to achieve very high accuracy, but because the solution phase is repeated for each cycle, it is desirable to hold the number of cycles to a minimum. Because the nodes from the previous mesh remain fixed for each cycle, selective refinement is ideally suited for iterative solution methods. The solution values obtained from the previous cycle, combined with interpolated values for the new degrees-of-freedom, provide an excellent initial guess for the next cycle [Ref. 2].

The major disadvantage is that the limited amount of refinement which can be performed in each cycle may necessitate several cycles to obtain an optimum grid. In addition, if new degrees-of-freedom require interelement continuity constraints, data management can become cumbersome unless the constraint is performed hierarchically.

D. SUBDOMAIN ISOLATION

One of the obvious disadvantages of the selective refinement technique is that the solution must be completely repeated for each cycle when, in fact, the number of degrees-of-freedom added in each cycle may be few in comparison to the total for the problem. In addition, the number of elements requiring refinement in each cycle may only account for a small portion of the domain. Although the refinement criterion has indicated where the grid is inadequate and the approximation is likely to be poor, the solution is repeated in each cycle for those nodes where the error is presumably small. Besides the apparent computational inefficiencies, this shortcoming severely restricts the amount of refinement which can be performed in the subregions of interest since it is desirable to confine the size of the problem within reasonable limits. An alternative approach is to reformulate the problem for those subregions where refinement is necessary and to accept the results of the initial analysis as an adequate solution for the remainder of the domain. The elements requiring further refinement, which constitute isolated subdomains of the original problem, can generally be subjected to significantly greater refinement than would otherwise be practical. The solution obtained from the initial analysis is then used in imposing boundary values on those degrees-of-freedom located on the boundaries of an individual subdomain. These can, in turn, be used to generate the boundary conditions for any additional boundary degrees-of-freedom introduced by the refinement using an appropriate interpolation scheme.

This grid optimization technique, which the author terms "subdomain isolation", has some further advantages over selective refinement. The subdomain may be selected arbitrarily small such that excellent results may be obtained

with a single cycle using uniform refinement. Therefore, the difficulties involving coarse-to-fine transition schemes, element elongation and interelement continuity can be avoided. Furthermore, one can choose as many subregions for refinement as desired without creating an excessively large problem.

The obstacle which may prevent this technique from being readily accepted is the notion that, by imposing erroneous boundary conditions on the subdomains, the convergence of the finite element method to the exact solution in these regions has somehow been tampered with. This aversion may be somewhat abated by considering a simple extension of Saint-Venant's Principle [Ref. 18: p. 33]. Although the conditions are not rigorously satisfied at the boundary, which may result in significant changes in the response locally, the effect at some sufficient distance away will be negligible. The numerical evidence supports this premise. While errors in the boundary values may somewhat restrict the accuracy of the dependent variable, great improvements can be realized in the accuracy of its gradients, which is more often the goal of the optimization. Since the initial analysis provides the boundary values for the subdomains, it is desirable that its solution be as accurate as reasonably possible. Fortunately, since subsequent refinements are not performed on the original grid, the initial analysis may involve significantly more degrees-of-freedom than in the case of selective refinement.

E. MESH GRADING

The final grid optimization technique to be discussed employs a mesh for which the element sizes are successively reduced, according to some geometric sequence, towards a selected region of the domain. One might argue that mesh

grading is not really an optimization technique since it is most often applied on an "a-priori" basis rather than adaptively, and that it does not lend itself well to the iterative application of a solution-based criterion. However, the technique is simple to use and its implementation requires few software modifications. Furthermore, a solution-based refinement criterion can be used to give a measure of the quality of the mesh to indicate whether a more pronounced grading may prove beneficial. Depending on the solution parameter of interest, mesh grading can provide excellent accuracy at a low analysis cost. This refinement method must therefore be considered among the grid optimization techniques.

For the less elaborate finite element preprocessors, mesh grading is often the only refinement means available without resorting to a uniformly finer mesh involving many more degrees-of-freedom. The most common method of implementation in two-dimensions is to first define the problem domain in terms of a curvilinear quadrilateral by selecting four keynodes along the problem boundary. Then the boundary nodes are spaced according to some geometric sequence based on the user-provided bias parameters which determine the density of the nodes towards selected points on the four quadrilateral edges. Finally, curves are generated to connect the boundary nodes on opposite edges of the quadrilateral, thus producing a graded mesh. This process, which can also be extended to three-dimensions, is the mesh generation scheme employed in the finite element code GIFTS [Ref. 19].

The major disadvantage of mesh grading is that in order to achieve sufficiently small elements in the region of interest, the elements must grow successively larger away from that region. This may be very undesirable, especially if refinement is called for in more than one region of the

domain, in which case the mesh must be generated and graded by subdomains, thereby complicating the data management involved.

Another disadvantage is that unless the domain possesses some special geometric symmetry, excessive element

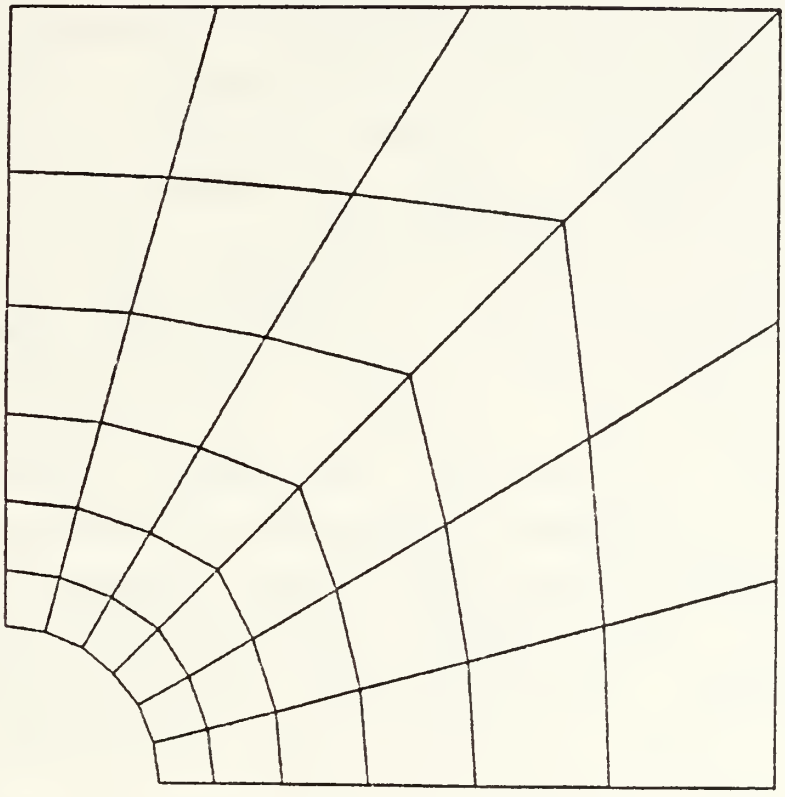


Figure 4.2 Graded Mesh for a Perforated Square Plate.

elongation will usually result if a highly pronounced grading is required. Some configurations are particularly well suited for refinement for mesh grading such as the classical perforated square plate problem in solid mechanics shown in Figure 4.2.

V. APPLICATION TO ONE-DIMENSIONAL PROBLEMS

Now that the necessary tools for performing grid optimization have been introduced, it is time to employ them in an attempt to obtain optimal solutions to some practical problems in engineering. An obvious starting point for such an investigation is the one-dimensional boundary value problem. While most of the fruitful research in grid optimization has concentrated on problems of higher dimensions, the one-dimensional problem is a very convenient device for studying finite element grid optimization. Foremost, one-dimensional finite element models possess a unique connectivity in that adjacent elements meet at their end nodal points. Therefore, refinement by the h- or p-versions, or by relocating nodal points becomes a trivial task, which does not involve any of the difficulties so frequently encountered with higher dimensional problems, such as preserving interelement continuity and maintaining optimal element shapes. Furthermore, one-dimensional studies can often provide valuable insight to the solution of more difficult higher dimensional problems.

The primary concerns in the selection of the problems to be studied were as follows:

(1) there should exist an analytical solution to provide a means of reliable error analysis;

(2) the solution and its resultants should exhibit sufficiently high gradients so that the effectiveness of the grid optimization is readily observed.

Because of the complexity and a certain degree of arbitrariness involved in the computation of element residuals and a-posteriori error estimates, the solution parameter variation is the refinement criterion of choice. There are several solution parameters which are easily computed, requiring minimal software changes to an existing finite element code.

Furthermore, for the one-dimensional investigation, it was decided to simplify the analysis by exploiting the linear elements. While it is granted that improved solution accuracy may generally be obtained by employing higher order elements, it will be assumed that conclusions based on the use of linear elements can be applied as well to elements of higher polynomial order.

A. ELASTIC CABLE PROBLEM

Consider an elastic cable under tension T, stretched between two points a distance 2L apart. If the cable is supported by a Winkler, or elastic, foundation of modulus k, and a concentrated load P is applied at the midspan, the resulting deflection $v(x)$, $(0 \leq x \leq L)$, is as shown in Figure 5.1. The analytical solution and finite element approach for this problem are presented in Appendix A.

The initial finite element analysis was performed using ten linear elements of uniform length. From this initial analysis the approximate distribution of the following solution resultants was obtained over the domain:

- the displacement, v (the solution)
- the slope, v'
- the strain energy, U
- the strain energy density, SED (dU/dx)

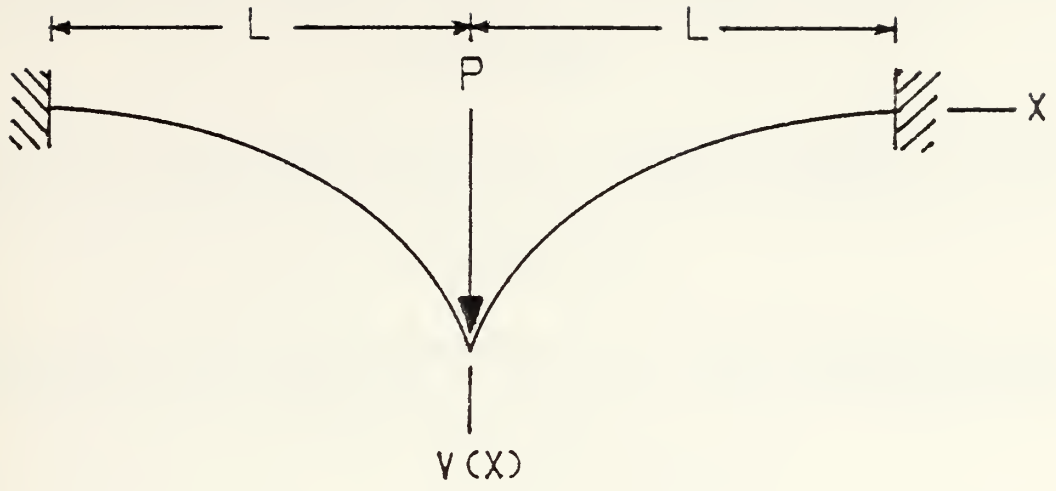
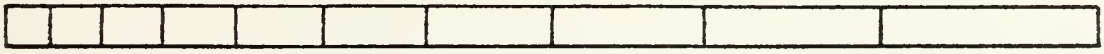


Figure 5.1 Tension Cable Deflection on Elastic Foundation.

Subsequent analyses were performed for finite element models using the same number of elements, but with the nodes redistributed to achieve approximately uniform variation of the above parameters over each element. Note that the strain energy refinement criterion produces elements of identical strain energy content. In addition, the problem was solved employing graded element models of various adjacent element length ratios. The resulting element models based on these refinement criteria are shown in Figure 5.2 (a-f). The graded model (b) for an element length ratio of 1.2 is presented for comparison because it produced good overall solution results.

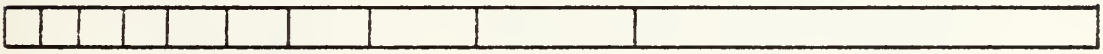
As previously mentioned, the solution resultants of primary interest are the maximum displacement, the maximum slope, and the integral of the displacement over the domain, because they represent important analogous solution results in nearly all fields in which finite element analysis is often performed. The accuracy obtained in these values for

(a) Uniform



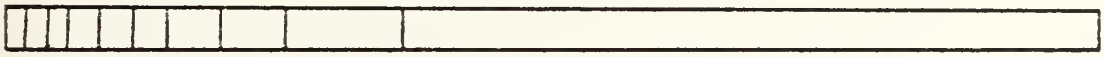
(b) Graded 1.2

(c) v



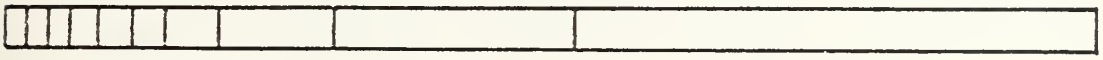
(d) v'

(e) U



(f) SED

(g) U_{mod}



(h) SED_{mod}

Figure 5.2 Tension Cable Refinements.

each of the refined models is presented in Table I. As can be seen in Figure 5.2, the strain energy and strain energy density criteria produced extreme variations of element length while the criteria of displacement and slope result

TABLE I

Tension Cable Problem Solution Results

Problem Parameters: L = 100 in
k = 1 psi
T = 1000 lb
P = 1000 lb

Variation Refinement Criterion	Percentage Relative Error v(max)	v'(max)	$\int_0^L v\,dx$
Uniform	-0.40	0.36	0.12
Graded (1.2)	-0.19	0.07	0.17
v	-0.18	0.06	0.39
v'	-0.23	0.05	0.87
U	-1.03	0.05	3.58
SED	-1.29	0.05	4.17
U (mod)	-0.53	0.04	1.63
SED (mod)	-0.51	0.03	1.48

in more moderate variations. It can be observed in Table I that the more pronounced refinements based on energy distribution result in greater accuracy for the maximum slope but with the accompanying severe penalty of significantly poorer estimations of the maximum displacement and the integral quantity. For this particular problem the uniform grid provides optimal accuracy of the integral quantity,

therefore refinement cannot reduce its error. Yet great improvement in the accuracy of the maximum slope and modest improvement in the accuracy of the maximum displacement can be achieved with moderate refinements based on the displacement and slope distributions.

One might assume, and correctly so, that the ability of the energy refinements to produce the best accuracy for the maximum slope is due to the extremely small elements which result in the area where that quantity occurs. Furthermore, it would be correct to propose that the reason for these refinements producing poorer estimates than the uniform model for the other two quantities of interest is that the excessively large elements in the regions of low gradients severely overstiffen the model there. It would then seem plausible tc improve the accuracy for the maximum displacement and the integral quantity by redistributing the nodes in these regions to prevent such excessively large elements. This was done by arbitrarily employing a grading scheme to the three largest elements to produce the modified refinements based on strain energy and strain energy density shown in Figure 5.2 (g) and (h). As can be seen in Table I, such a modification did indeed significantly reduce the errors in the maximum displacement and the integral, but it even further improved the accuracy for the maximum slope.

One might conclude from Table I that the best overall model was obtained using the graded mesh, and that since it is easier to obtain, it should be deemed the optimal grid. But this particular grading was chosen for precisely that reason and was presented only as a means of comparison. In practice, the selection of a grading ratio is somewhat arbitrary and making an adequate choice may be difficult.

There is justifiable confusion as to which refinement produced the "best" solution accuracy for this problem and it raises perhaps the most important issue in the subject of

grid refinement. Before any optimization process can be pursued, the optimization goals must be explicitly defined. Clearly, as is the case in this problem, the designation of the optimum grid would depend heavily upon which of the three solution resultants is most critical to the analysis.

B. TAPERED BAR PROBLEM

The linearly tapered bar under axial loading has received considerable attention and was one of the early problems for which analytical grid optimization was employed. Consider a tapered elastic bar of length L and modulus E, fixed at one end, with an axial load P applied at the other, for which the axial displacement u(x), $(0 \leq x \leq L)$, is desired. The cross-sectional area varies linearly from A_o at the fixed end to A_t at the tip, as shown

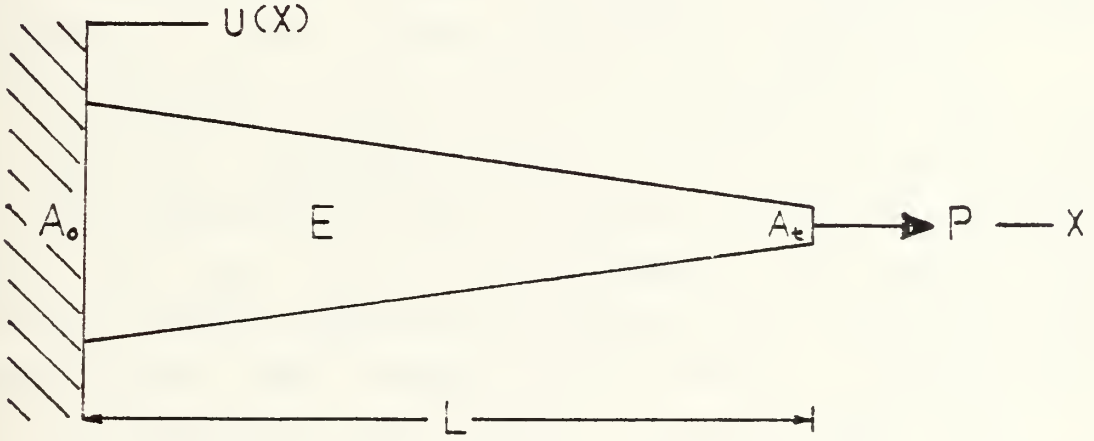


Figure 5.3 Linearly Tapered Bar Under Axial Loading.

in Figure 5.3. The analytical solution and finite element approach are presented in Appendix B.

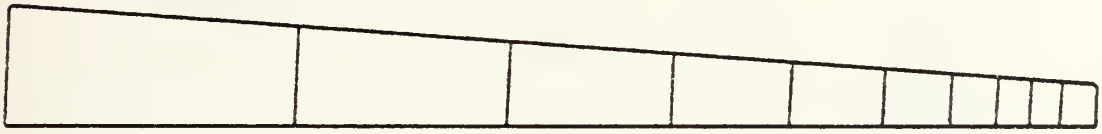
One of the significant features of the tapered bar problem is that the maximum stress can be very difficult to model accurately, and it is for precisely these problems exhibiting large strain gradients that grid optimization becomes most beneficial. Interestingly, the stresses obtained at the element midpoints are exact for this problem, and the difficulty arises from the inability of the constant slope shape functions to model the maximum stress occurring at the boundary. In examining this problem, Prager [Ref. 20] demonstrated analytically that when each element has the same strain energy content, the relative error in displacement is identical for all the nodes. However, this phenomenon appears peculiar to this problem and the author does not subscribe to such a measure of an optimum grid. Judging the effectiveness of a particular refinement based upon the deviation or the mean value of the pointwise errors generally tends to be unfavorable to optimization procedures since they almost always introduce many more nodes in those regions where the response is most difficult to model. Hence, an improved solution may have a larger mean value of the pointwise errors [Ref. 3].

The criteria employed in the refinement of the tapered bar model are identical to those used in the cable problem and their effects are shown in Figure 5.4 (a-e). Two exceptions are that now the displacement and strain energy criteria produce identical refinements, and the graded model chosen as the best overall is now based on a grading ratio of 1.4, producing a more drastic refinement than that of 1.2 for the cable. This further demonstrates the difficulty involved in obtaining adequate element grids on an "a-priori" basis.

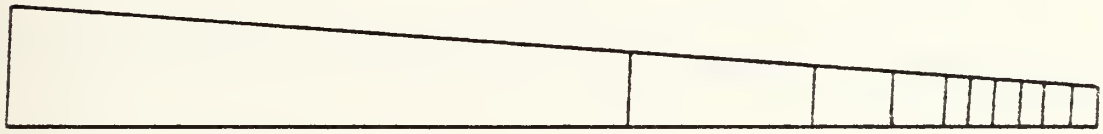
The solution results are presented in Table II and the most readily apparent observation for this problem is the large errors in the maximum slope, which would severely

(a) Uniform

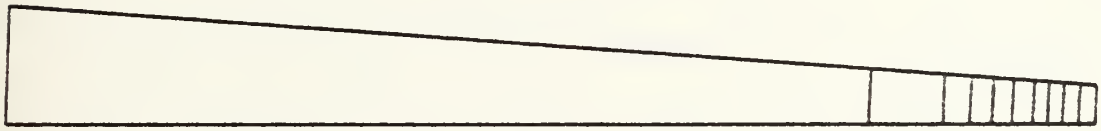
(b) Graded 1.4



(c) u and U



(d) u'



(e) SED

(f) SED_{mod}

Figure 5.4 Tapered Bar Refinements.

TABLE II

Tapered Bar Problem Solution Results

Problem Parameters: $L = 100$ in, $A = 10.5$ in^2, $A = 0.5$ in^2, $E = 10 \times 10^6$ psi, $P = 10 \times 10^3$ lb

Variation Refinement Criterion	Percentage Relative Error u(max)	u'(max)	$\int_0^L u\,dx$
Uniform	-3.80	-37.5	0.68
Graded (1.4)	-0.78	-4.1	0.14
u ; U	-0.85	-10.6	0.15
u'	-1.81	-7.7	0.33
SED	-6.54	-3.6	1.18
SED (mod)	-1.99	-3.6	0.36

underestimate the maximum stress. These results are based on quadratic extrapolation of the exact slopes at the element midpoints, since the linear shape functions would produce even poorer estimations of the maximum slope. As before, the more extreme refinement based on the strain energy density variation provides the most accurate estimation of the maximum slope, but with the accompanying degradation in estimates for maximum displacement and the integral of displacement. Again, the large errors in these values may be significantly reduced by employing a grading scheme to restrict the size of the larger elements as shown in Figure 5.4 (f). Unlike the previous problem, such a modification has no effect on the estimate of maximum slope because of the extrapolation of the element midpoint slopes, which are exact regardless of the element model.

A different version of the tapered bar problem, for which the displacement and strain energy criteria will not produce identical refinements, involves replacing the concentrated tip load P with a linearly varying axially distributed load q(x), specified by the values at the fixed end q_o and the tip q_t. The problem may be further modified by reversing the bar such that the maximum slope occurs at the fixed end, while the maximum displacement occurs at the

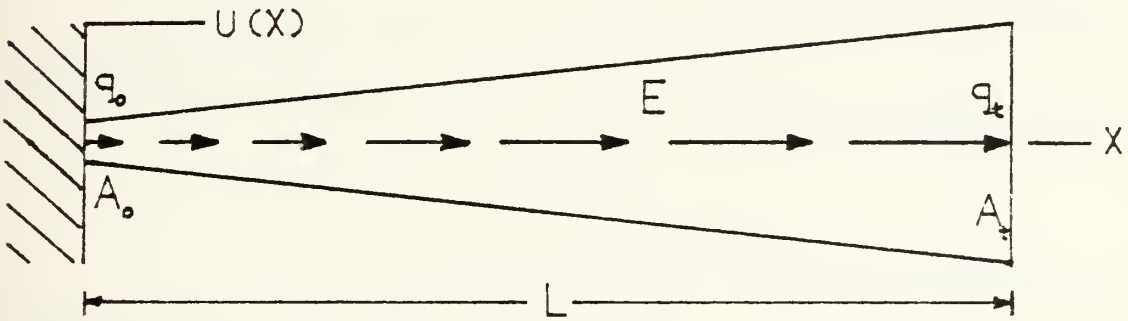


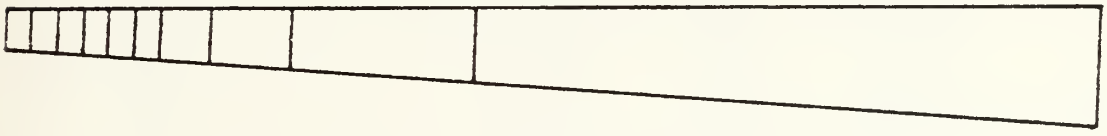
Figure 5.5 Reversed Tapered Bar with Distributed Load.

free end as shown in Figure 5.5. The case of the linearly varying distributed load is included in the formulation in Appendix B.

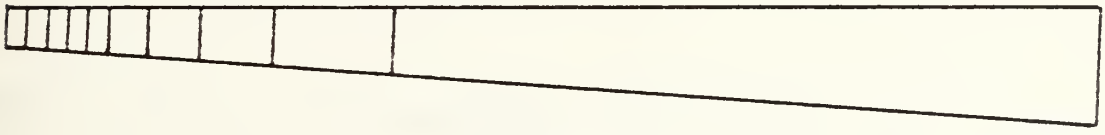
This problem was solved for a uniformly distributed load using the same procedure as in the previous two problems. The refinement models are presented in Figure 5.6 and the solution results in Table III. The observations are consistent with those made in the previous problems, but now one would likely agree that the refinement based on the strain

(a) Uniform

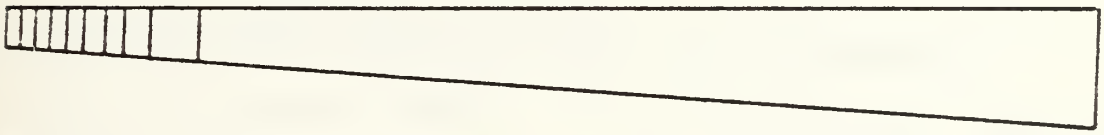
(b) u



(c) u'



(d) U



(e) SED

(f) SED_{mod}

Figure 5.6 Reversed Tapered Bar Refinements.

TABLE III

Reversed Tapered Bar Solution Results

Problem Parameters: $L = 100$ in, $A = 0.5$ in^2, $A = 10.5$ in^2, $E = 10 \times 10^6$ psi, $q = 100$ lb/in

Variation Refinement Criterion	Percentage Relative Error $u(max)$	$u'(max)$	$\int_0^L u\,dx$
Uniform	-5.5	-39.4	-7.1
u	-2.0	-7.5	-2.6
u'	-2.7	-8.1	-3.4
U	-3.7	-5.9	-4.7
SED	-11.9	-3.4	-15.3
SED (mod)	-3.1	-3.4	-4.0

energy density would represent an optimal grid, provided that modifications are introduced to prevent any elements from growing excessively large.

C. GUIDELINES FOR ONE-DIMENSIONAL GRID OPTIMIZATION

The most important lesson to be learned from this one-dimensional study is that the grid optimization procedure is necessarily dictated by the optimization goal, or the underlying purpose for performing the finite element analysis. No element grid can possibly provide optimum accuracy for every solution resultant of interest. In solving these simple problems, a balance has been sought for achieving adequate accuracy for three of the more important solution

resultants, with emphasis on the maximum value of the derivative of the dependent variable, which more often is not only the most important part of the solution but also the most difficult to obtain accurately in finite element analysis.

The important grid optimization techniques of introducing more degrees-of-freedom by subdividing the elements or increasing their polynomial order have been intentionally omitted in favor of the optimization strategy of seeking maximum solution accuracy for a specified number of degrees-of-freedom using linear elements. This is because such a procedure is not so straightforward in two-dimensional problems where the number of degrees-of-freedom are dependent on some geometric considerations, which do not appear in problems of one-dimension. Based on this choice of optimization strategy, it appears the strain energy density variation provides the most useful criterion for the adaptive refinement of the initial grid. Yet all three problems demonstrated some pathological results that can arise when the elements are permitted to grow excessively large in the regions where the strain energy density varies the least. In applying a scheme to restrict the size of the largest elements, no mention has been made of how to determine when an element is excessively large. It has become the experience of the author that any element representing over half of the domain should probably be considered too large, and measures should be employed to restrict its size.

It would appear, at least for these classes of problems, that this difficulty of decreasing accuracy of a particular solution parameter for successive refinements can be ignored by merely accepting the largest value among the cycles as the most accurate solution result. For example, it was demonstrated that the refinement based on strain energy density provided significant improvement in the accuracy for

the maximum slope but underestimated the maximum displacement even more than the initial uniform grid. Assuming that the linear element model always underestimates such maxima, the maximum slope for refined grid and the maximum displacement for the unrefined grid could be accepted as the optimal results of the analysis. The fallacies of such a procedure are that, first, the refinement may not represent the optimal grid as it has been defined and, second, for self-adaptive finite element codes the user is provided with the "optimum grid" of the final cycle and the solution results thereof.

Melosh and Marcal [Ref. 21] have proposed an alternative use of the refinement criterion based on strain energy density variation which avoids the problem of excessive element growth altogether. Beginning with a reasonably coarse uniform grid, those elements with the greatest strain energy density variation are selectively refined by either subdividing them or increasing their polynomial order with the introduction of additional degrees-of-freedom. While such a procedure does not equi-distribute the element strain energy variations, it can reduce them all to some prespecified tolerance, such as a percentage of the average element variations for the initial analysis. Because this procedure is particularly attractive for grid refinement in problems of higher dimensions, it will be employed extensively for the study of grid optimization for two-dimensional problems in the next chapter.

VI. APPLICATION TO TWO-DIMENSIONAL PROBLEMS

Since investigators began working in the field of finite element grid optimization in the early 1970's, nearly all of the effort has been devoted to the development of a systematic procedure for obtaining optimal grids for two-dimensional problems of elasticity. Even today there are several competing approaches to this problem and no particular one has yet been overwhelmingly accepted as the preferred method of grid optimization. While it is the two-dimensional problem for which most of these techniques have been developed, their application to such can be much more difficult than for the one-dimensional case. Almost invariably when performing grid refinement on two-dimensional domains, the analyst is confronted with the problems of maintaining interelement compatibility and preventing severe element distortion.

In selecting an appropriate two-dimensional problem for the application of some grid optimization techniques and a comparison of their effectiveness, it is desirable that the test case possess the following properties:

- the analytical solution should exist in order to perform reliable error analysis
- the solution should exhibit sufficiently large gradients to provide a meaningful measure of the refinement effectiveness
- the idealization should have one degree-of-freedom per node and possess simple boundary conditions to minimize the computational effort involved in repeated solutions

There are few problems that meet these criteria, but Saint-Venant torsion of a non-circular section provides a good test case.

A. PROBLEM DESCRIPTION

Consider a solid circular shaft of radius "a" made from isotropic material of shear modulus G and having a circular groove, or keyway, of radius b along a generator of the

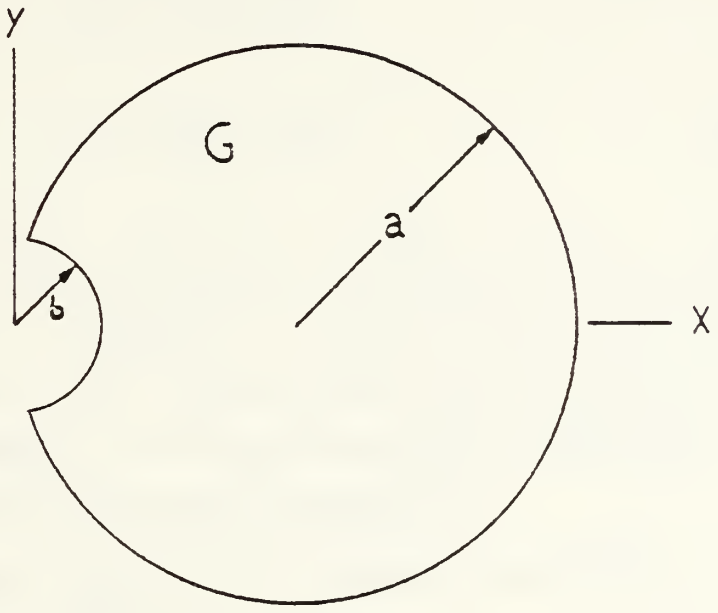


Figure 6.1 Cross-section of Shaft with Keyway.

shaft. The shaft cross-section is shown in Figure 6.1. The shaft is subjected to an applied torque T which produces an angle of twist per unit length θ. The problem may be solved by finding the Prandtl torsional stress function ψ which satisfies the governing differential equation:

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + 2 = 0 \qquad \text{(Eqn. 6.1)}$$

subject to the Dirichlet condition that $\psi = 0$ on the section boundary. The torsional stress function is defined such that the shear stress τ at any point on the domain may be expressed as:

$$\tau = G\theta \left[\left(\frac{\partial \psi}{\partial x} \right)^2 + \left(\frac{\partial \psi}{\partial y} \right)^2 \right]^{\frac{1}{2}} \qquad \text{(Eqn. 6.2)}$$

For this formulation, the angle of twist θ is prescribed, rather than the applied torque T. The torque is is calculated from the area integral:

$$T = 2G\theta \int_A \psi \, dA \qquad \text{(Eqn. 6.3)}$$

The analytical solutions of Equations 6.1 and 6.2 are derived by Sokolnikoff [Ref. 22: pp. 141-143] and are presented in Appendix C along with the evaluation of Equation 6.3 and a prescribed finite element formulation. For this problem, the three solution resultants of interest for the grid optimization study are:

(1) maximum value of the dependent variable, or torsion function ψ_{max};

(2) maximum value of the gradient of the dependent variable (a quantity proportional to maximum shearing stress τ_{max});

(3) the area integral of the dependent variable over the domain (a quantity proportional to the applied torque T).

These quantities - the dependent variable, its gradient, and an integral thereof - are selected as representative of entities whose error one might wish to minimize in a finite element analysis.

B. COMPUTER IMPLEMENTATION

As can be seen in Figure 6.1, the domain of this problem is symmetric about the x-axis, therefore the finite element solution need only be obtained for the upper half of the domain. For all of the solutions presented herein the problem geometry is defined by assigning the dimensionless ratio, b/a = 0.4, and an acceptable upper limit on the analysis cost was arbitrarily chosen to be that corresponding to approximately one hundred nodal points. The computation and assembly of the finite element matrices and solution of the resulting system of equations was performed using the steady state heat conduction operations of CAL-NPS [Ref. 23]. This group of subroutines comprises an efficient finite element code for solving Poisson's equation in two or three dimensions and has the additional advantage of permitting variable-noded isoparametric elements.

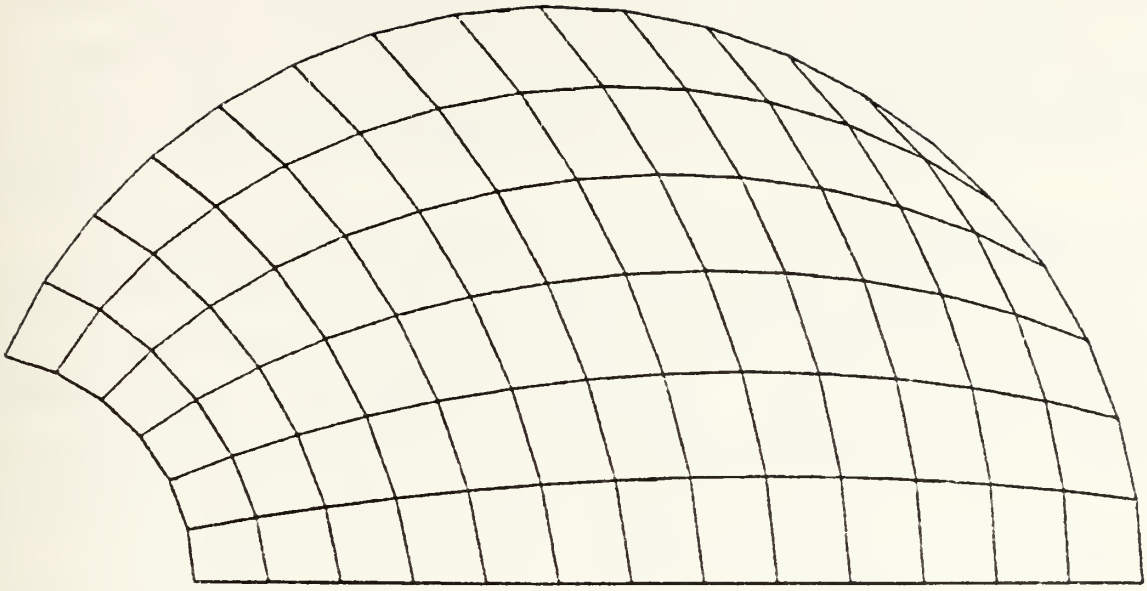
Since there was no readily available interactive preprocessor which lent itself well to adaptive mesh refinement, the author had no choice but to create his own. Since the problem domain is simply connected, the automatic mesh generation was performed employing inverse mapping of a single cubic isoparametric element of the serendipity family onto the problem domain [Ref. 24: pp. 228-229]. Mapped boundary nodes were repositioned to conform to the actual domain boundary and additional nodes generated during the refinement process were mapped using the same procedure.

Since the finite element code selected for this investigation provided output only for the nodal values of the dependent variable, it was coupled to the author's postprocessor. Such a postprocessor is necessary in the optimization process for computing nodal values of shear stresses and strain energy density, element contributions to torque and total strain energy, and exact results from theory.

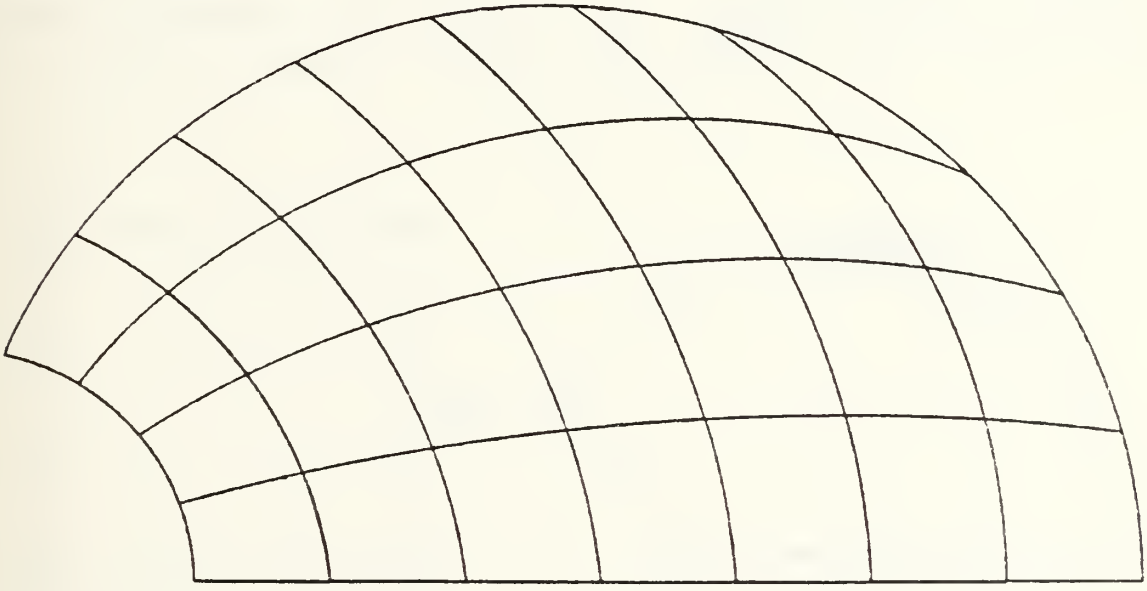
C. ASYMPTOTIC ERROR ANALYSIS FOR UNIFORM REFINEMENT

The concept of asymptotic convergence rate for uniformly refined grids was presented in Chapter 3. When the number of uniformly distributed degrees-of-freedom is sufficiently large, the log-log plot of the relative energy error versus the number of degrees-of-freedom is approximately linear in the asymptotic range. The slope of this line represents the asymptotic rate of convergence in energy.

It so happens that relative error in the torque T of this problem is equal to the relative energy error and therefore exhibits this linear asymptotic behavior on the log-log plot against the number of uniformly distributed degrees-of-freedom. Fortunately, the other two solution resultants of interest behave similarly. This will prove very beneficial in performing the error analysis for this two-dimensional study for two reasons. First, because it is unnecessarily difficult to construct an optimal grid with the same number of degrees-of-freedom as a uniform grid, the linear behavior of the solution resultants in the asymptotic range on the log-log scale permits interpolation for any number of degrees-of-freedom. Then the solution results for a uniform grid of the identical number of degrees-of-freedom provides a reference for comparison to determine the effectiveness of the optimization technique. Secondly, if the convergence rate of a particular solution resultant is extremely slow, as is often the case for maximum stress, it becomes difficult to gain an appreciation of the true effectiveness of the optimization. For example, an order of magnitude reduction in the relative solution error may require an order of magnitude increase in the number of degrees-of-freedom using uniform refinement, but relatively few additional degrees-of-freedom using an optimization technique. Therefore, it will be enlightening to extrapolate



(a) Linear Element Grid



(b) Quadratic Element Grid

Figure 6.2 Uniform Linear and Quadratic Element Grids.

the relative error versus degrees-of-freedom curve to obtain a rough approximation of the number of degrees-of-freedom necessary to obtain solution accuracies similar to the optimal grid, but using successively finer uniform grids. Of course, this is only an estimation and ignores such realities as numerical ill-conditioning and computer round-off error.

A uniform grid is one for which all of the elements are of the same size h and the same polynomial order p. Clearly, it is impossible to obtain such a grid for this particular domain using isoparametric mapping, but a nearly uniform grid may be constructed in which the elements are of approximately the same size. Such uniform grids are shown for the cases of linear quadrilateral elements and quadratic serendipity elements (Fig. 6.2). For this geometry, the uniform grid is not uniquely defined for a specified number of elements. This is because, in performing isoparametric mapping, there must be specified four keynodes on the actual domain boundary to correspond to the four corner nodes of the parent square. Since this domain has only three

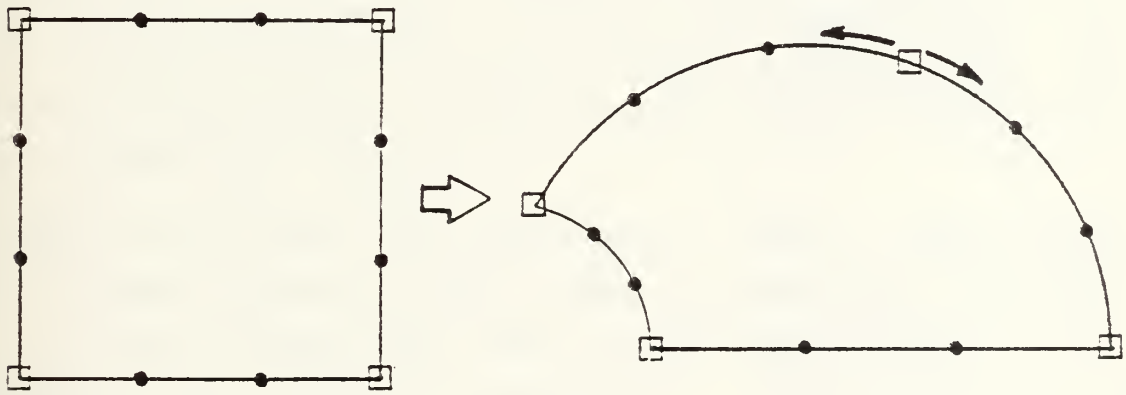


Figure 6.3 Keynode Placement for Isoparametric Mapping.

vertices, the placement of the fourth keynode is at the discretion of the analyst (Fig. 6.3), and can have a noticeable effect on the solution results.

The asymptotic error analysis was performed for the three solution resultants of interest using uniform grids of linear and quadratic elements. The results are presented in Figure 6.4. All of the solution resultants behaved as predicted with the exception of the maximum torsion function value using linear elements, $\psi_{m(1)}$. It appears that the accuracy of this particular parameter is very strongly dependent upon the keynode placement. The curve constructed in Figure 6.4 represents an average for several keynode positions.

While Figure 6.4 is intended primarily to serve as a reference tool for future analyses, it provides some interesting information:

(1) For the cases of maximum torsion function value and applied torque (and energy), the asymptotic rate of convergence using quadratic elements is more than twice that for linear elements.

(2) While the error in torque for the quadratic case is always smaller than that for the linear case, the linear grid may provide better accuracy for the maximum torsion function value ψ_{max} in the pre-asymptotic range.

(3) Both accuracy and convergence rate in the maximum shear stress are only minutely greater for the quadratic element grid than for the linear one.

However, for this last observation, the point must be made that for the linear element grid, the maximum shear stress was obtained by quadratic interpolation rather than from the linear shape functions. While this will greatly improve the

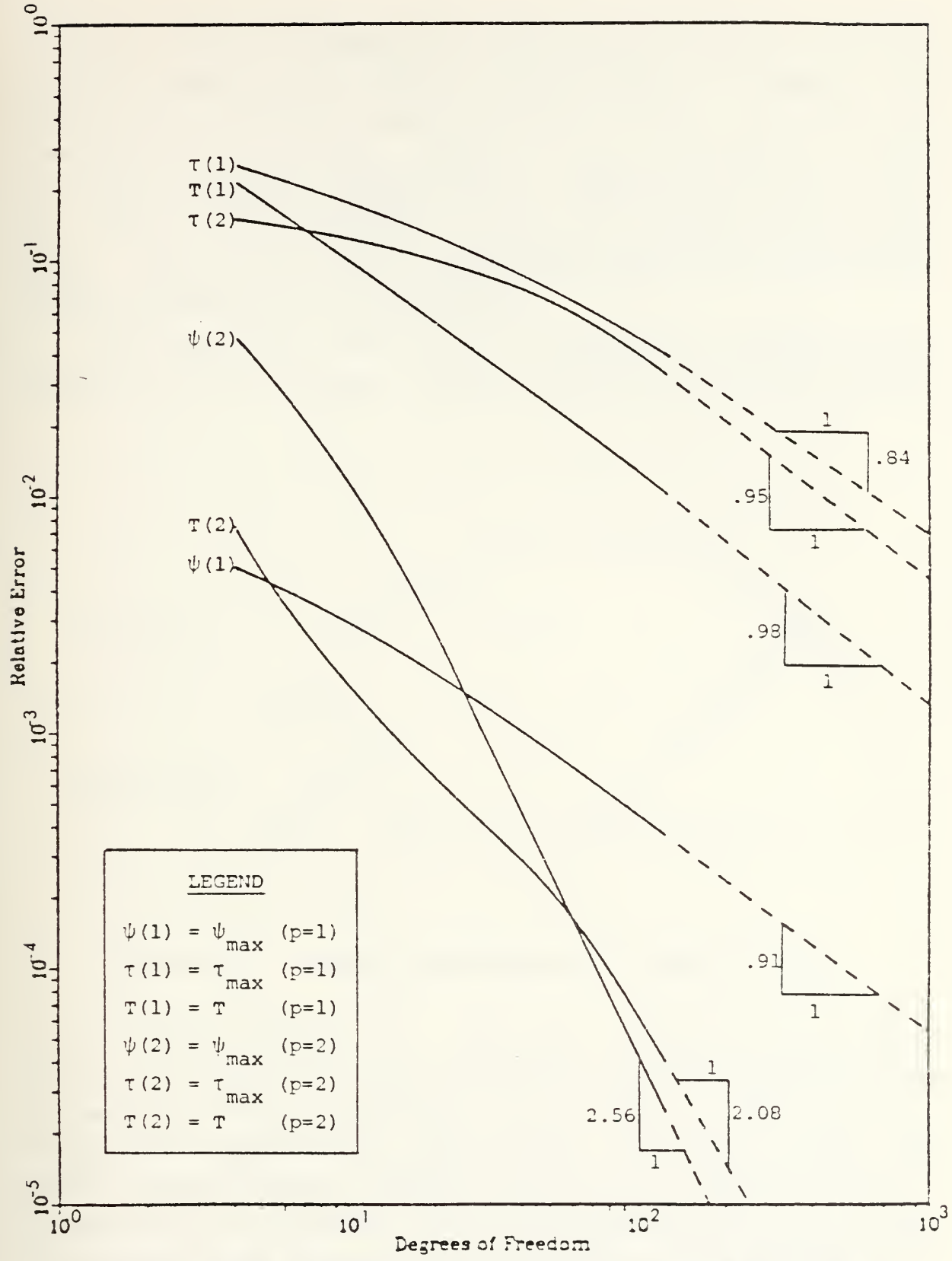


Figure 6.4 Asymptotic Convergence for Uniform Refinement.

accuracy of the maximum shear stress approximation, it will have no effect on its rate of convergence. Therefore, if obtaining an optimal estimate of the maximum shear was the purpose of the analysis, there is much to be said on behalf of linear elements besides their computational efficiency. Of course, this observation is based on uniform grid refinement, which would rarely compete favorably with the optimization techniques to be examined.

The reason that the rate of convergence in maximum shear stress is so poor using uniform refinement for this problem

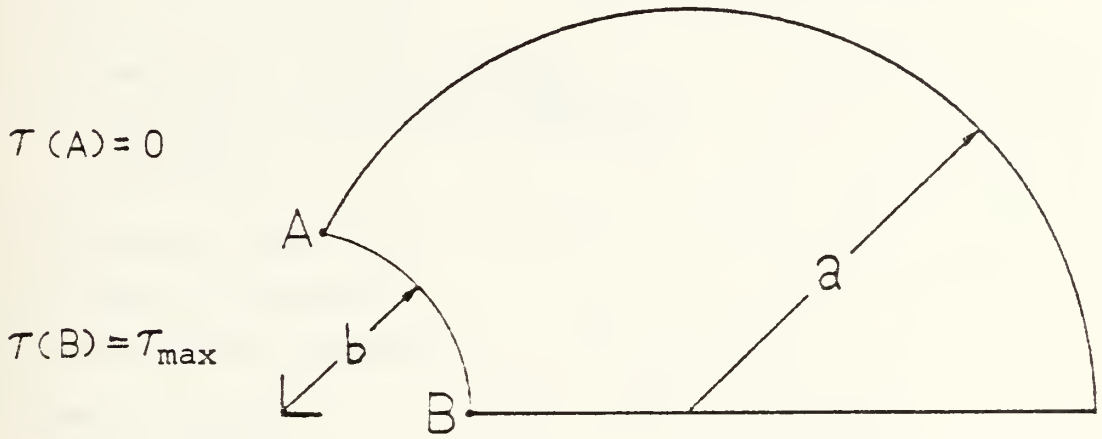


Figure 6.5 Stress Distribution on Shaft Keyway.

can be seen in Figure 6.5. The shear stress varies greatly over a short distance, by increasing from zero at point A to its maximum value at point B. As a result, there exists a region of excessively large strain gradients along the keyway which severely hinders the rate of convergence when a uniform grid is employed. If the keyway radius were allowed to approach zero producing a singularity in the solution, it

would likely be necessary to employ even higher order elements via the p-version in order to achieve convergence using uniform refinement [Ref. 1].

D. PROBLEM SOLUTION WITH GRID OPTIMIZATION

The finite element solution of the torsion problem will be obtained employing the following grid optimization techniques as presented in Chapter 4:

- Contouring
  - contours of the torsion function; linear elements
  - contours of shear stress; linear elements
  - contours of strain energy density; linear elements
- Selective Refinement
  - h-version; linear elements
  - h-version; quadratic elements
  - p-version
- Subdomain Isolation
  - linear elements
  - quadratic elements
- Mesh Grading
  - linear elements
  - quadratic elements

1. Contouring

The original finite element analysis was performed on a uniform grid of 98 linear elements, 78 nodes, and 72 degrees-of-freedom. The finite element solution provided the nodal values of the torsion function ψ, from which the conventional nodal resultants of shear stress τ, and applied torque T were computed. Based upon the maximum and

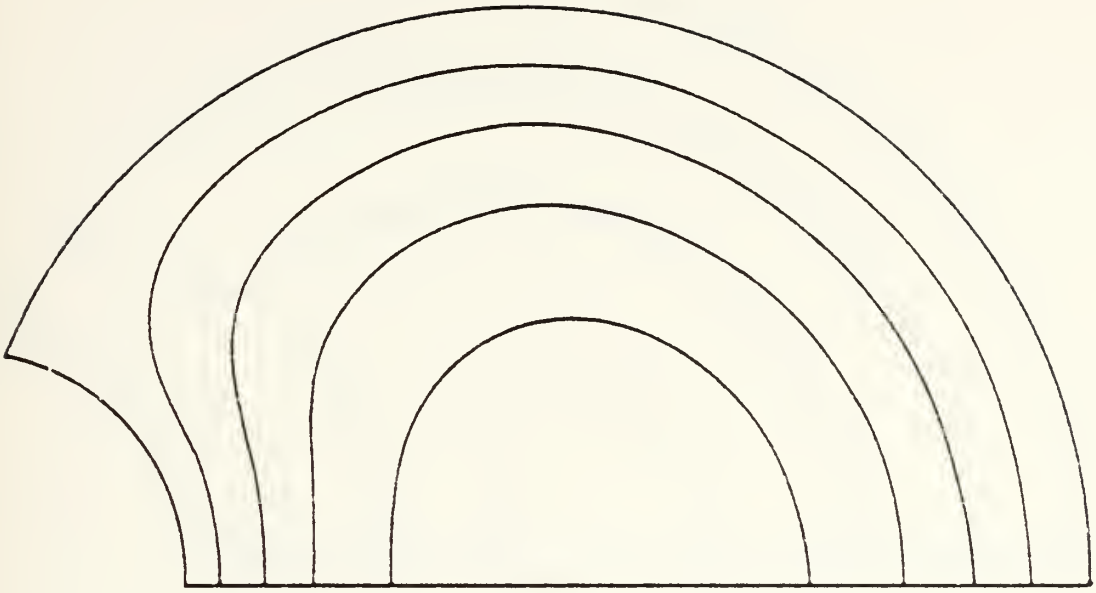
minimum values obtained for each parameter, along with consideration for their values along the boundary, the contours to be used for nodal placement in each case were selected. The number of contours for each case was chosen to maintain approximately the same number of degrees-of-freedom as for the initial analysis.

The points for each contour value selected were obtained by linearly interpolating between the nodal values of each parameter obtained from the initial analysis. The contours were constructed by smoothly connecting the points by hand. The element layout along the contours posed the most formidable problem because the coarse-to-fine transition often resulted in severe element distortion, and it sometimes became necessary to degenerate quadrilateral elements into triangles when the transition was acute. It was decided that the optimal element shapes should be preserved along the contours in regions of highest stress.

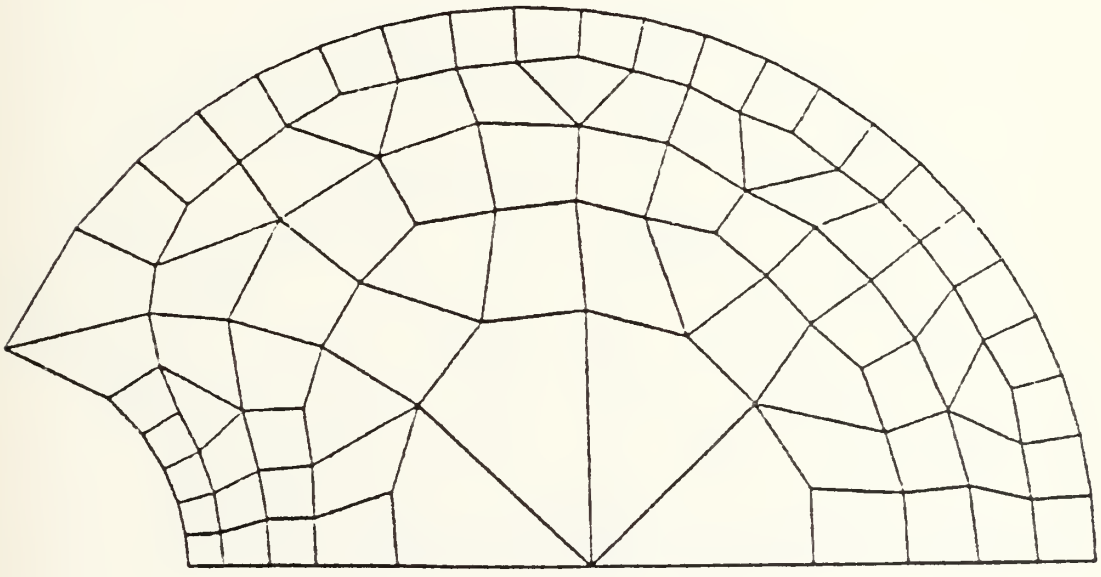
The contours obtained and the corresponding grid are presented for each of the following solution resultants:

- torsion function (Fig. 6.6)
- shear stress (Fig. 6.7)
- strain energy density, SED (Fig. 6.8)

The resulting grid for each of the response function contours produces smaller elements in the region of greatest stress near the bottom of the keyway and around the periphery of the shaft where the stress is moderately high. Consequently, the elements near the center where the stress is zero are larger. These, of course, are the desired effects for an optimization criterion. A somewhat unusual behavior is observed at the point of intersection of the keyway and the shaft boundary where the stress is also zero. Apparently, the shear stress gradient is larger than the gradients in torsion function and the strain energy density,

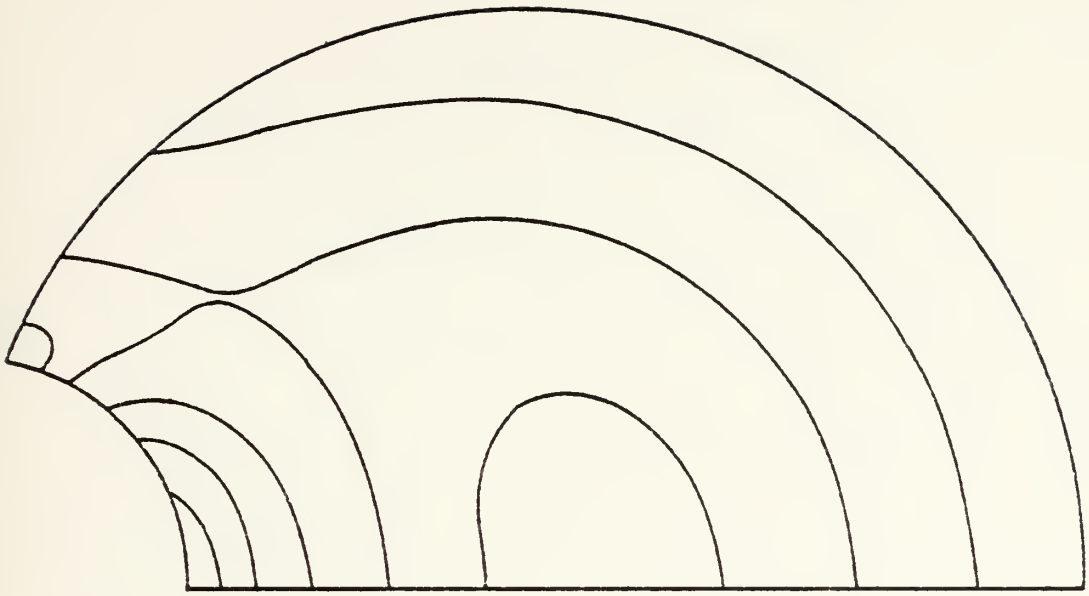


(a) Contours

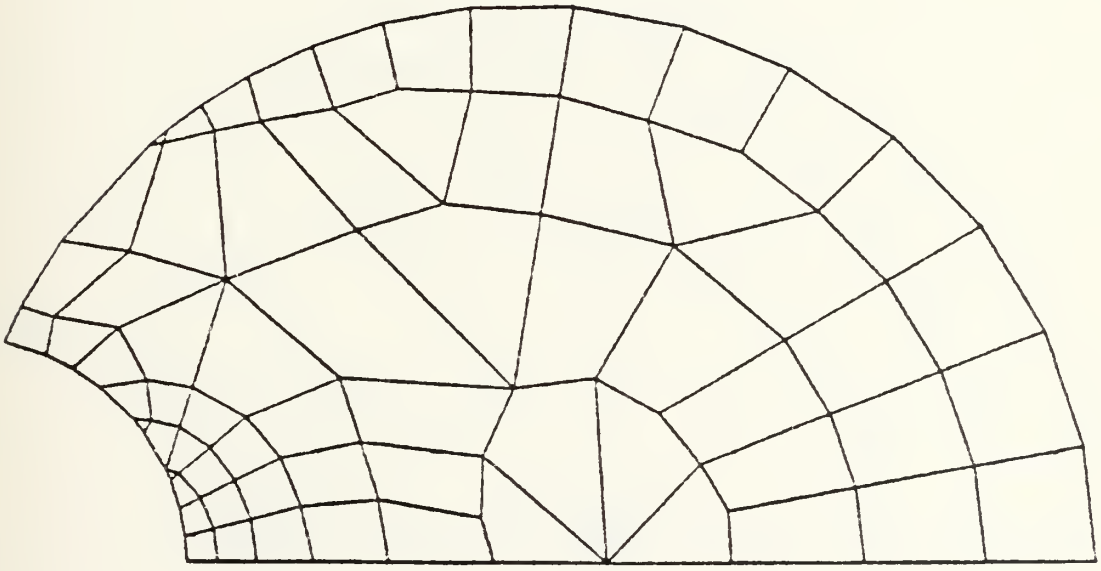


(b) Corresponding Grid

Figure 6.6 Contouring for the Torsion Function.

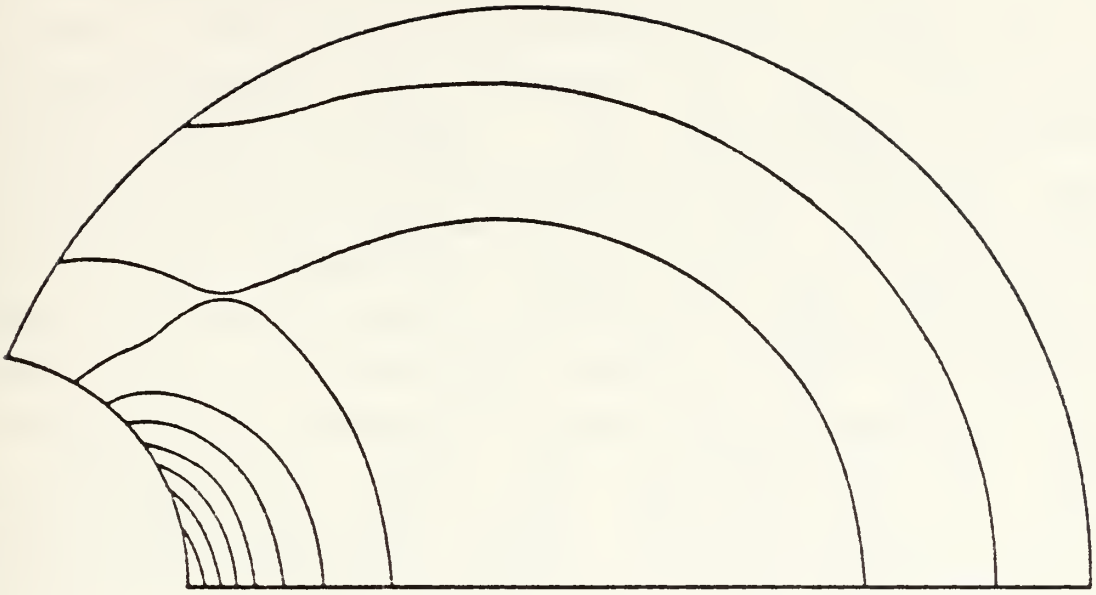


(a) Contours

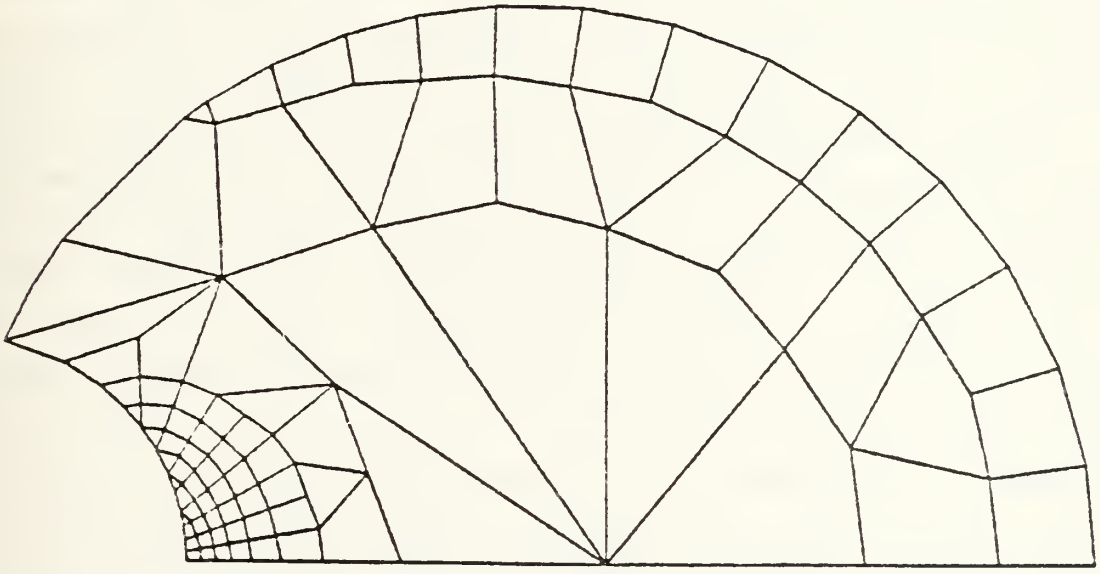


(b) Corresponding Grid

Figure 6.7 Contouring for the Shear Stress Function.



(a) Contours



(b) Corresponding Grid

Figure 6.8 Contouring for the SED Function.

resulting in smaller elements being produced in that region by the shear stress criterion (Fig. 6.7) than by the other criteria. While all of the grids possess to some extent the desirable features of an optimal grid, the strain energy density function produces a far more drastic refinement towards the point of maximum stress, while the others represent more moderate refinements. In fact, the SED contours are so dense around the keyway that the coarse-to-fine element transition scheme must include degenerate quadrilaterals to avoid violating the contours. Note also that the coarse-to-fine transition for the torsion function response is fairly smooth whereas the transition for the strain energy density and shear stress refinement tends to produce distorted and elongated elements. This is aggravated by the fact that, unlike the torsion function, the shear stress and strain energy density are not monotonic over the domain.

The solution results obtained for each grid are presented in the upper half of Table IV. At first glance, the results of the refinements are disappointing in comparison to the parenthetic values obtained using a uniform grid. While all three criteria produce improved accuracy for the maximum shear stress, the errors for the maximum torsion function value grow extremely large. Recalling the observations made for the one-dimensional study, it would follow that such behavior is probably due to the unusually large elements near the center of the domain. The entries in the lower half of Table IV reflect the drastic improvement obtained by simply introducing a few additional degrees-of-freedom along those element edges which grew exceptionally long during the optimization process, thus increasing their polynomial order from one to two. Not only did this modification reduce the large errors for the maximum torsion function estimation, but modest improvements were also obtained in the estimations of the other resultants as well.

TABLE IV

Contouring Solution Results

Contours	No. Elem	No. Node	No. DOF	Percentage Relative Error[1] ψ_{max}	τ_{max} / Gθ	T / Gθ
ψ	82	102	73	5.0 (0.06)	-1.51 (-6.00)	-1.88 (-1.76)
τ	66	84	56	2.4 (0.08)	-2.40 (-7.20)	-2.81 (-2.22)
SED	82	100	70	10.6 (0.06)	-3.87 (-6.20)	-4.41 (-1.82)
ψ (mod)	82	110	79	-0.29 (0.06)	-1.24 (-5.70)	-1.19 (-1.65)
τ (mod)	66	89	61	-0.11 (0.07)	-2.28 (-6.82)	-2.62 (-2.06)
SED (mod)	82	108	77	0.28 (0.06)	-0.65 (-5.80)	-1.96 (-1.66)

[1] Parenthetic values for uniform linear grid of same number of degrees-of-freedcm.

Once again, the selection of the optimum grid would depend predominantly on the optimization goal of the analysis, but one would likely agree that the strain energy density variation along with some modification to restrict excessive element growth provides the superior refinement criterion.

An additional word of caution is in order for the contouring techniques for grid optimization. Because the problem must be completely redefined from scratch after the initial analysis, the preprocessing cost can become enormous, especially if several cycles are employed to obtain more precise contours as some authors suggest. Unless there is available an interactive automatic mesh generator based on this technique, such as the one described in Reference [Ref. 11], contouring should be abandoned in favor of some more easily implemented grid optimization techniques employing similar refinement criteria.

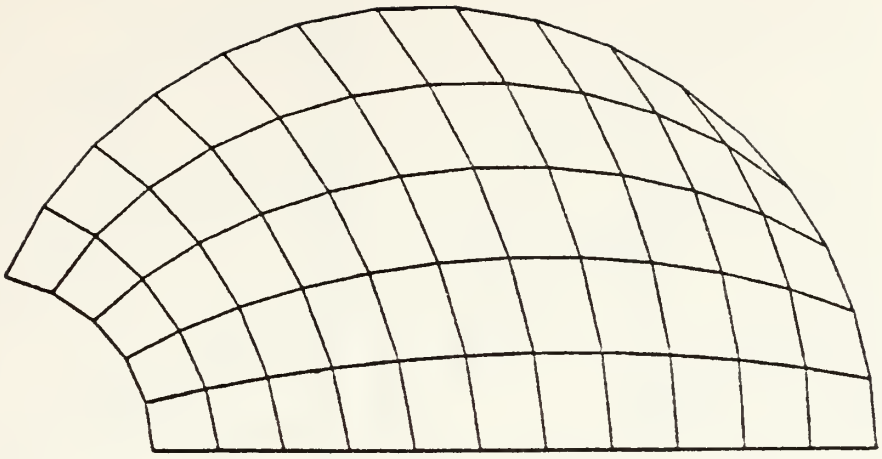
2. Selective Refinement

The simplest way to avoid the problems encountered in the contouring techniques is to perform the initial analysis on a reasonably coarse grid and then to selectively refine those elements over which the strain energy density varies the most. The critical concern then arises as to how coarse the initial grid should be. If the preprocessor employs the necessary constraint conditions to permit the "father-to-four-sons" element subdivision scheme directly, or if hierarchical refinement is employed, then the initial grid should be just coarse enough to adequately define the problem and to limit the number of refinement cycles necessary. The latter becomes even less of a concern if iterative solvers are employed. If, on the other hand, coarse-to-fine transition schemes are used to implement the h-version or only low polynomial order elements are available in the p-version, then the initial grid must be sufficiently fine

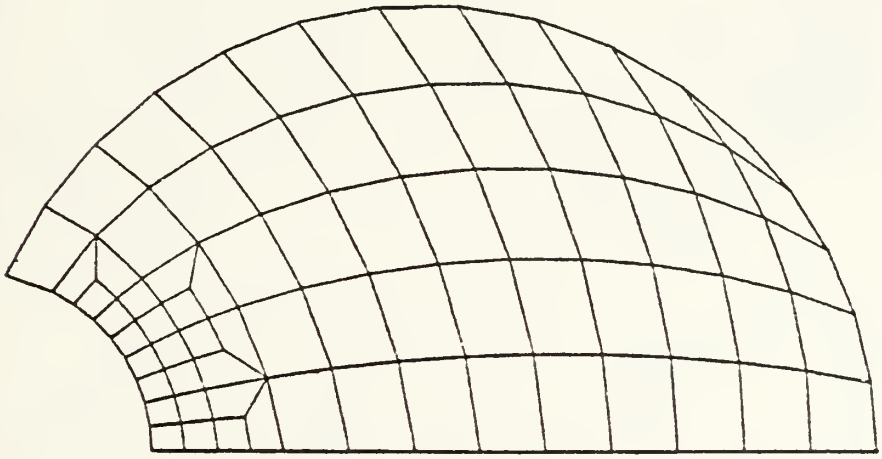
so as not to restrict severely the amount of refinement which can be performed in any given cycle. Unfortunately, the conditions under which this investigation was conducted were those of the latter.

a. The h-Version

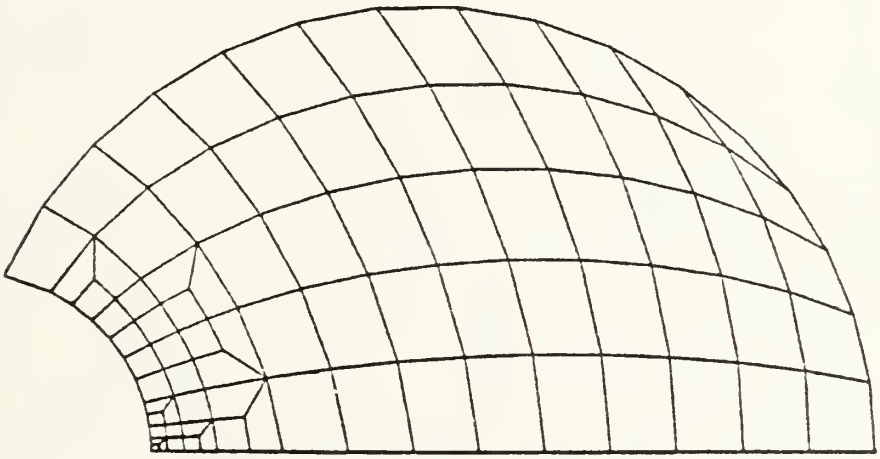
Selective refinement by the h-version was performed on both linear and quadratic element grids. For the linear case, the initial analysis was performed on a uniform grid of 55 elements, 72 nodes, and 50 degrees-of-freedom. The initial quadratic analysis employed an eight element uniform grid of 37 nodes and 20 degrees-of-freedom. The reason for such a great disparity in the number of elements for the initial analyses is that subdividing a quadratic element introduces many more degrees-of-freedom than the subdivision of a linear element. These numbers were chosen to provide approximately the same number of degrees-of-freedom for the optimum grid of the final cycle for each case. The initial analysis is performed and those elements over which the strain energy density variation is significantly greater become candidates for refinement. The refinement is performed by subdividing each candidate element into four new ones by constructing a coarse-to-fine transition zone of "buffer" elements around the refined region. Successive analyses and selective refinements are repeated until the maximum element strain energy density variation is approximately that of the remainder of the grid. The process is further improved when the nodal values of the strain energy density are used to indicate the general direction in which the refinement is to proceed. This permits multiple refinements in the same cycle, thereby reducing the number of cycles required to arrive at the optimal grid. For this problem, the linear grid required two refinement cycles while the quadratic grid required three cycles. The



(a) Initial Grid



(b) Cycle 1



(c) Cycle 2

Figure 6.9 Selective Refinement Procedure - Linear Elements

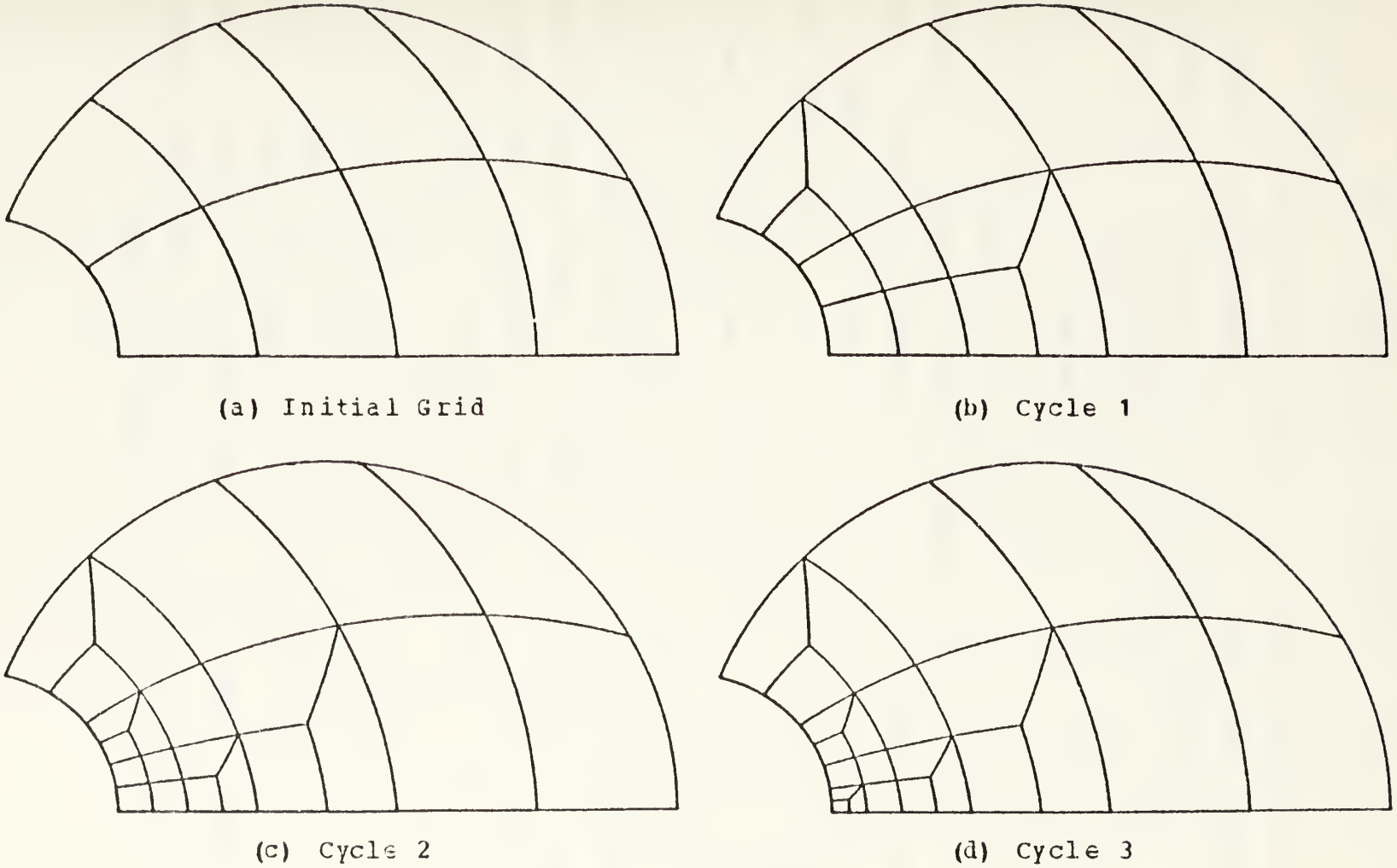


Figure 6.10 Selective Refinement Procedure - Quadratic Elements.

selective refinement process is depicted in Figures 6.9 and 6.10 for the linear and quadratic element grid, respectively.

The solution results for each selective refinement cycle are presented in Table V. The most impressive observation to be made is the significant improvement in the maximum shear stress estimate for successive cycles. While there is also modest improvement in the accuracy of the torque estimate for successive cycles, when compared to the estimate obtained from the uniform grid of the same number of degrees-of-freedom and polynomial order, the refinement estimate of torque is slightly poorer. This is because additional degrees-of-freedom are being introduced in only a small region of the domain but the torque, and energy, are global quantities. The author has no satisfactory explanation why the estimate for the maximum torsion function improves for successive refinements of the linear grid but not for the quadratic case. However, as has already been mentioned, this particular solution parameter appears very sensitive to such problem variables as nodal placements and element shapes; hence, its behavior is difficult to predict, even when the refinement is applied to regions remote from the point where the maximum torsion function value occurs, as was the case in these examples. For computational reasons, it is desirable to restrict the number of refinement cycles to a necessary minimum. In this example, the quadratic grid required an additional cycle over the linear grid but this is because it is necessary to perform the initial quadratic analysis using far fewer degrees-of-freedom. Therefore, the early cycles of the quadratic analysis actually represent comparatively smaller problems.

TABLE V

Selective Refinement Solution Results

Refinement Cycle	No. Elem	No. Node	No. DOF	Percentage Relative Error[1] ψ_{max}	τ_{max} / Gθ	T / Gθ
Linear Element Grid						
Initial	55	72	50	0.084	-7.67	-2.47
1	72	92	66	-0.022 (0.065)	-2.99 (-6.42)	-2.26 (-1.94)
2	81	104	75	-0.033 (0.058)	-0.90 (-5.92)	-2.27 (-1.72)
Quadratic Element Grid						
Initial	8	37	20	-0.29	-9.87	-0.066
1	15	62	41	-0.31 (0.04)	-4.17 (-7.53)	-0.041 (-0.028)
2	22	87	62	-0.31 (0.02)	-1.40 (-6.05)	-0.039 (-0.016)
3	26	103	74	-0.31 (0.01)	-0.14 (-5.34)	-0.039 (-0.012)

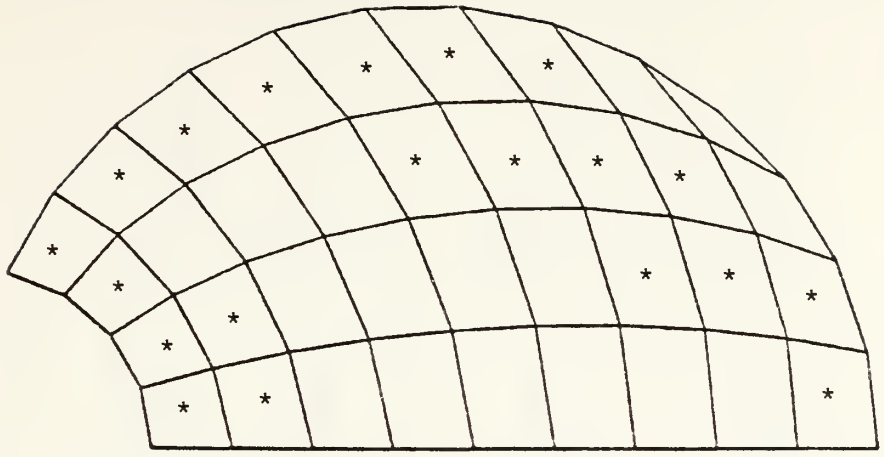
[1]Parenthetic values for uniform grid of same polynomial order and same number of degrees-of-freedom.

b. The p-Version

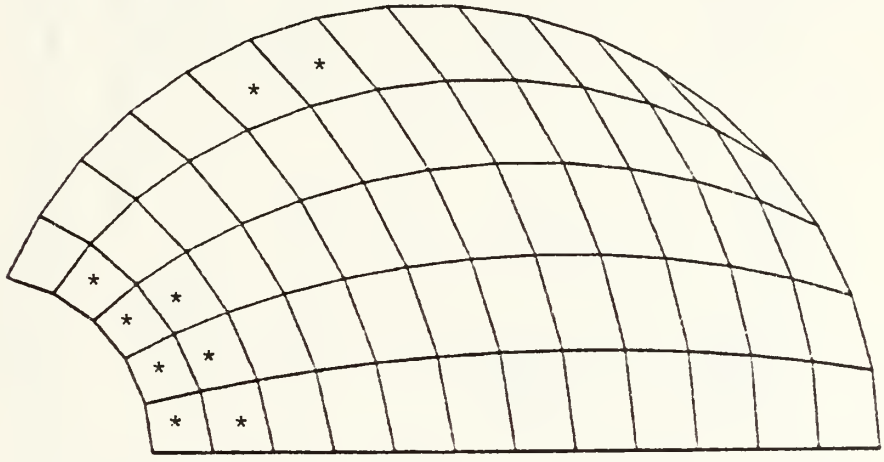
Before continuing to the next optimization technique, it is worthwhile to take a quick look at selective refinement employing the p-version. Because the finite element code used in this investigation only provided for element orders of one and two, the advantages of the method cannot be fully realized, but the effects of a single cycle can be examined.

Beginning with three uniform grids with differing numbers of linear elements, the initial analyses were performed. In each case, the elements over which the strain energy density varied the most were transformed from 4-noded Lagrangian elements into 8-noded serendipity elements by the addition of midside nodes. The element grids are shown in Figure 6.11 and the asterisks denote those elements for which the polynomial order was increased. In this example, the number of elements to be refined in each case was chosen so as to achieve approximately the same number of degrees-of-freedom after a single cycle.

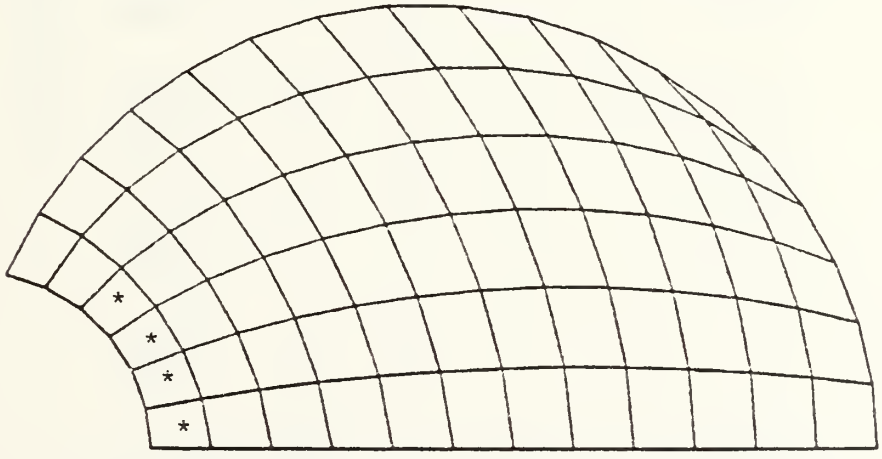
The solution results are shown in Table VI. Significant improvements in the estimate of the maximum shear stress were achieved for each case. An improvement in the estimated torque was also realized for all three cases, but was more noticeable when the number of refined elements was larger. This is because quadratic elements are far superior to linear elements in the modeling of integral quantities, as observed in Figure 6.4. Somewhat disturbing is the increased error in the estimate of the maximum torsion function value observed in two of the three refinements even though the elements in the vicinity of its occurrence were not affected. Again, this is likely attributable to the unusual behavior patterns of this quantity already discussed.



(a) 76 Degrees-of-Freedom - 20 Refined Elements



(b) 76 Degrees-of-Freedom - 9 Refined Elements



(c) 75 Degrees-of-Freedom - 4 Refined Elements

Figure 6.11 Selective Refinement Employing p-Version.

TABLE VI

Selective p-Version Solution Results

Grid	No. Refin	No. Elem	No. Node	No. DOF	Percentage Relative Error[1] ψ_{max}	τ_{max} / Gθ	T / Gθ
Initial		36	50	32	0.12	-10.0	-3.69
Refined	20	36	107	76	0.53 (0.06)	-3.73 (-5.85)	-1.29 (-1.71)
Initial		60	78	55	0.077	-7.27	-2.27
Refined	9	60	105	76	-0.064 (0.057)	-2.67 (-5.85)	-1.91 (-1.71)
Initial		72	91	66	-0.065	-6.42	-1.94
Refined	4	72	104	75	-0.134 (0.058)	-2.42 (-5.92)	-1.82 (-1.72)

[1]Parenthetic values for uniform linear grid of same number of degrees-of-freedom.

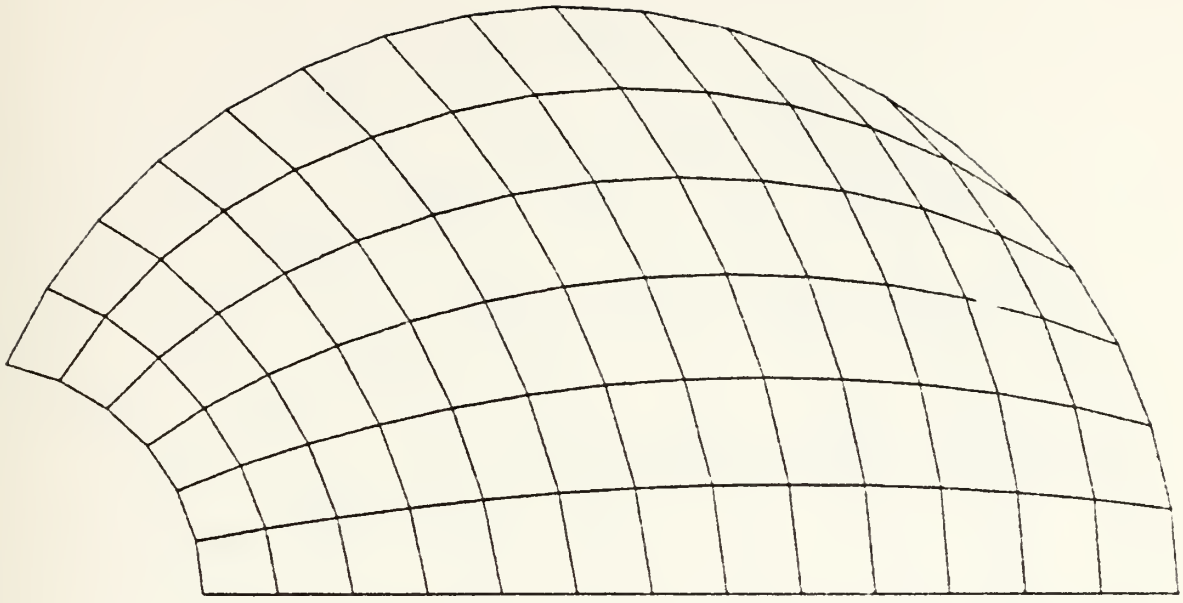
In order to perform additional cycles of the p-version, it would be necessary to alter the refinement criterion slightly. Because the element sizes do not change for successive cycles, the need for refinement would necessarily be based on strain energy density variation between nodes rather than over the elements.

Selective refinement employing the p-version is most efficiently implemented hierarchically, in which case it acquires some attractive computational advantages. It is unfortunate that time did not permit further investigation here, but the need for future research is evident.

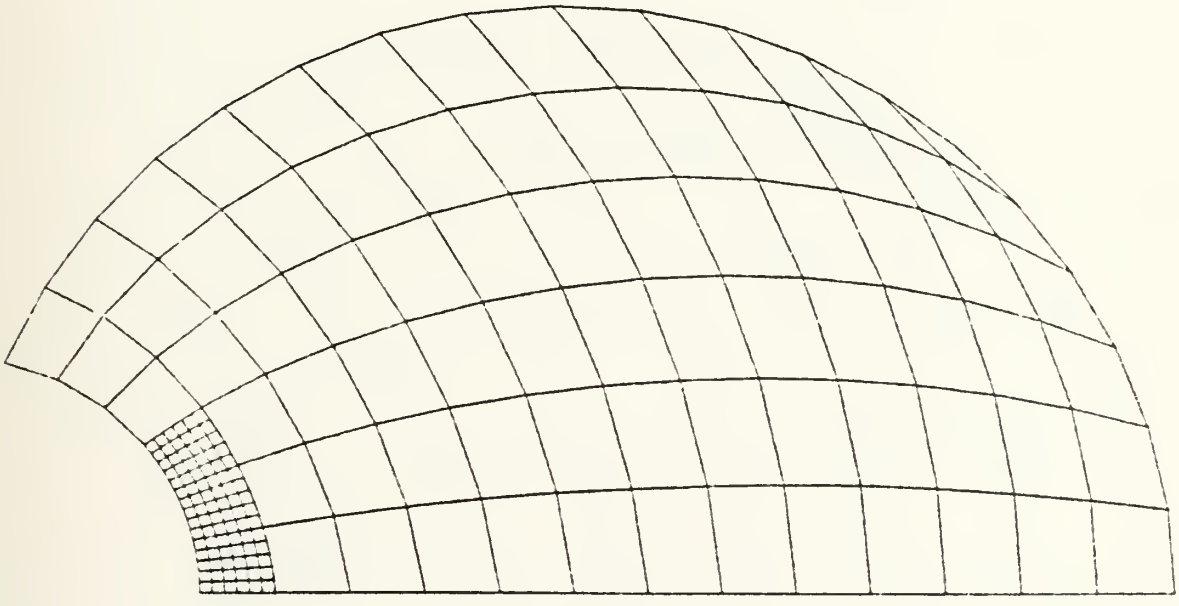
3. Subdomain Isolation

The refinement criterion and initial procedures in employing subdomain isolation are identical to those used in selective refinement. After the candidate elements for refinement are identified, they are completely isolated from the remainder of the domain and solved as smaller subdomain problems. The advantages of the technique are twofold. By isolating the elements to be refined the solution is not repeated in each cycle for those elements for which the initial analysis is assumed adequate. Furthermore, by eliminating most of the degrees-of-freedom over the entire domain, the subsequent refinement of the isolated region can be much greater than would otherwise be practical.

As before, the technique was applied to both linear and quadratic uniform element grids. Those elements of the initial analysis over which the strain energy density variation was exceptionally large were isolated to comprise the subdomain in each case. There were three such elements of the initial linear grid and two for the quadratic grid. Each subdomain was uniformly refined to achieve approximately the same number of degrees-of-freedom as the initial analyses. The process is depicted in Figure 6.12 for the linear grid and Figure 6.13 for the quadratic case.

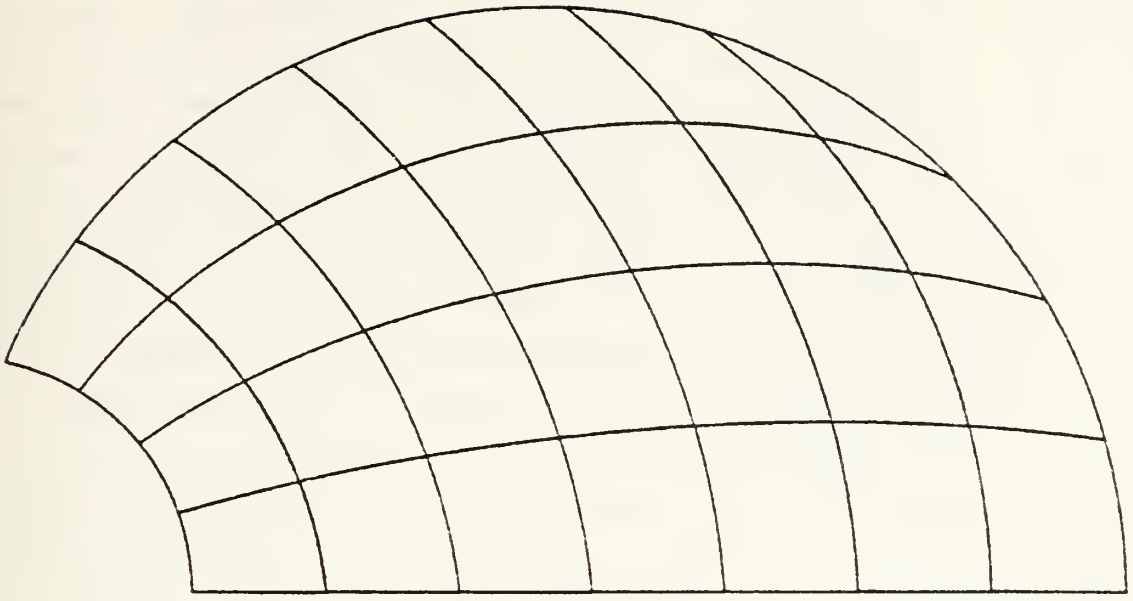


(a) Initial Grid

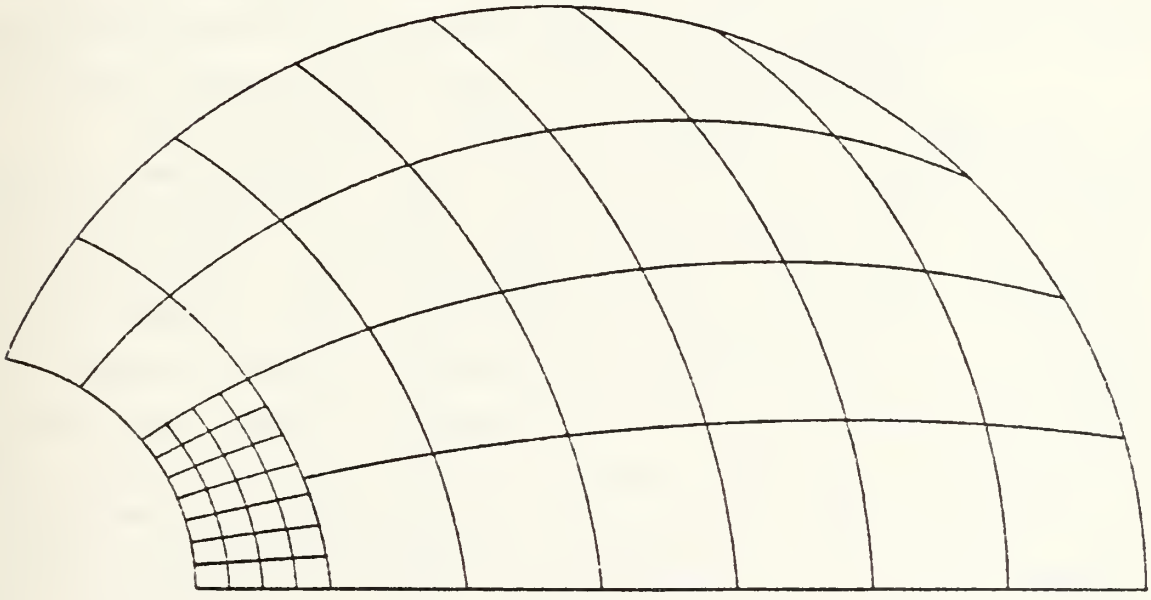


(b) Refinement of Isolated Subdomain

Figure 6.12 Subdomain Isolation - Linear Elements.



(a) Initial Grid



(b) Refinement of Isolated Subdomain

Figure 6.13 Subdomain Isolation - Quadratic Elements.

In performing subdomain isolation the governing equations remain the same, while only the domain and the boundary conditions are altered. When the subdomain boundary has nodes common to the initial grid, then the boundary values for those nodes are simply the solution values obtained from the initial analysis. The boundary values arising from the introduction of new boundary nodes during the refinement process must be generated by interpolation of the solution results of the initial analysis. One of the options for an interpolation scheme is simply to use the shape functions of the unrefined elements. This may not be desirable in the case of linear elements, so a higher order interpolation may be employed. In this example a third order Lagrangian polynomial was convenient for the linear case since there are four nodes from the initial analysis along the right-hand boundary of the subdomain. Since there are only two such nodes on the upper boundary, it is necessary to "borrow" some adjacent nodal values from the discarded portion of the domain in the generation of new boundary values using higher order interpolation.

The solution results presented in Table VII are quite remarkable. In a single cycle, the solution accuracy for the maximum shear stress has increased by a full order of magnitude. No other optimization technique examined in this investigation produced such improvements. Note that the higher order polynomial interpolation for the boundary values did improve the solution results for the linear case. One of the disadvantages of this technique is that the refinement can produce no improvement in the estimation of local quantities outside the subdomain. As in this example, the estimation of the maximum torsion function value obtained from the initial analysis must be accepted as the optimum since it occurs outside the subdomain. Furthermore, since its value is predominantly affected by refinements in

TABLE VII

Subdomain Isolation Solution Results

Grid	No. Elem	No. Node	No. DOF	B.C. Poly. Ord.	Percentage Relative Error[1] ψ_{max}	τ_{max} / Gθ	T / Gθ
Linear Element Grid (3 elements isolated)							
Initial	78	98	72		0.060	-6.06	-1.77
Subdomain	90	112	75	1	0.060 (0.058)	-0.91 (-5.92)	-1.59 (-1.72)
				3	0.060 (0.058)	-0.61 (-5.92)	-1.59 (-1.72)
Quadratic Element Grid (2 elements isolated)							
Initial	28	107	76		0.0093	-5.26	-0.012
Subdomain	28	107	70	2	0.0093 (.0114)	-0.53 (-5.60)	-0.009 (-.013)

[1]Parenthetic values for uniform grid of same polynomial order and same number of degrees-of-freedom.

regions of high gradients, it is doubtful that isolating a new subdomain using the elements adjacent to its point of occurrence would noticeably improve its accuracy. However, since the torque is a globally computed quantity, refinement will improve the accuracy of its contribution from the subdomain resulting in improvement of its accuracy overall as observed in Table VII. It is this strictly local nature of the subdomain isolation technique which restricts its applicability. But if the optimization goals are well defined and it is understood under which conditions and for what parameters it is effective, it can be an extremely powerful grid optimization technique.

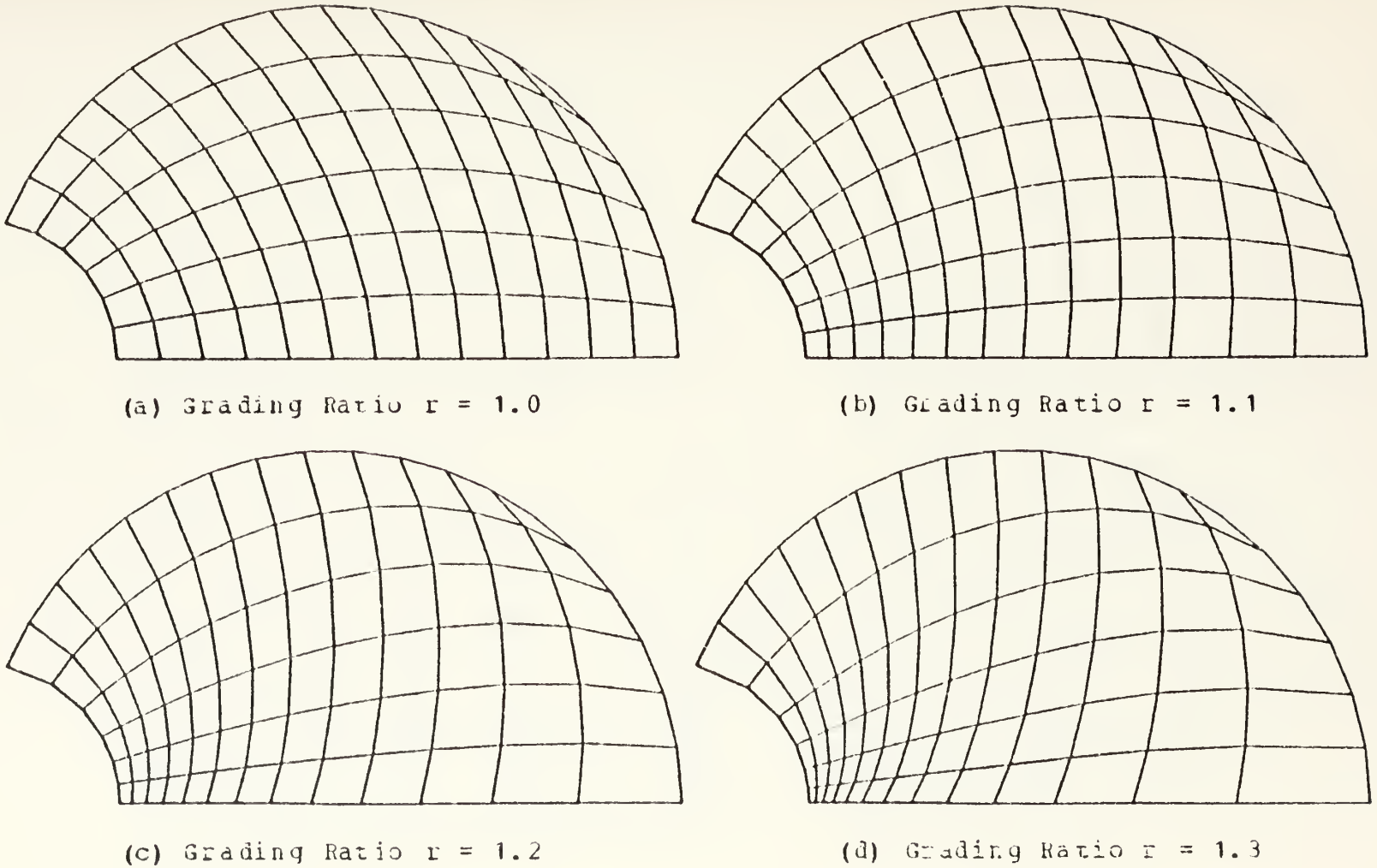
4. Mesh Grading

While mesh grading is nearly always performed on an "a-priori" basis, it may also be employed adaptively to provide a simple grid optimization technique. After an initial solution has been obtained, the analysis may be repeated using various combinations of grading ratios in order to achieve a more uniform distribution of the element strain energy density variations. Here the grading ratio refers to the constant ratio of adjacent element lengths along a boundary of the domain to which grading is applied. There are several drawbacks to the technique, the first being that a good combination of grading ratios may be difficult to obtain in a reasonable number of cycles. The other difficulty is that if smaller elements are desired in more than one region the domain must be refined and constructed by subregions.

Unfortunately, the domain of this problem is not well suited for mesh grading since it possesses no favorable geometric symmetry. Hence, the resulting element elongation and distortion would become severe for larger grading ratios. For simplicity, the nodal placements will be biased

only along the two domain boundaries adjacent to the point of maximum stress and the same grading ratio will be applied for each. This will result in small elements near the bottom of the keyway and large elements along the periphery of the shaft. While this is not the most desirable grid topology, it will produce a more uniform distribution of the element SED variations.

The technique was applied to both linear and quadratic element grids starting with a uniform mesh and successively increasing the grading ratio until the elements along the shaft periphery exhibited SED variations as large as those for the elements along the keyway. In both cases this condition occurred beyond the point where excessive element elongation would be expected to produce numerical inaccuracies. Graded meshes for selected grading ratios r, are shown in Figure 6.14 for linear elements and Figure 6.15 for quadratic elements. The solution results are presented in Table VIII. As can be observed, the maximum shear stress estimate improves for each successive increase in the grading ratio. However, the cost of such improvement is the accompanying degradation in the estimate of the maximum torsion function value. This is to be expected since the two maxima occur at different locations in the domain and therefore decreasing the size of the elements in the vicinity of one will necessarily increase the element size near the other. Note that this degradation is not nearly so severe for the quadratic elements, and the accuracy actually improves for a low value of the grading ratio. This is because the higher order interpolation can better accomodate larger elements. For both linear and quadratic element grids the optimal accuracy in torque estimation occurs for the moderately graded meshes.



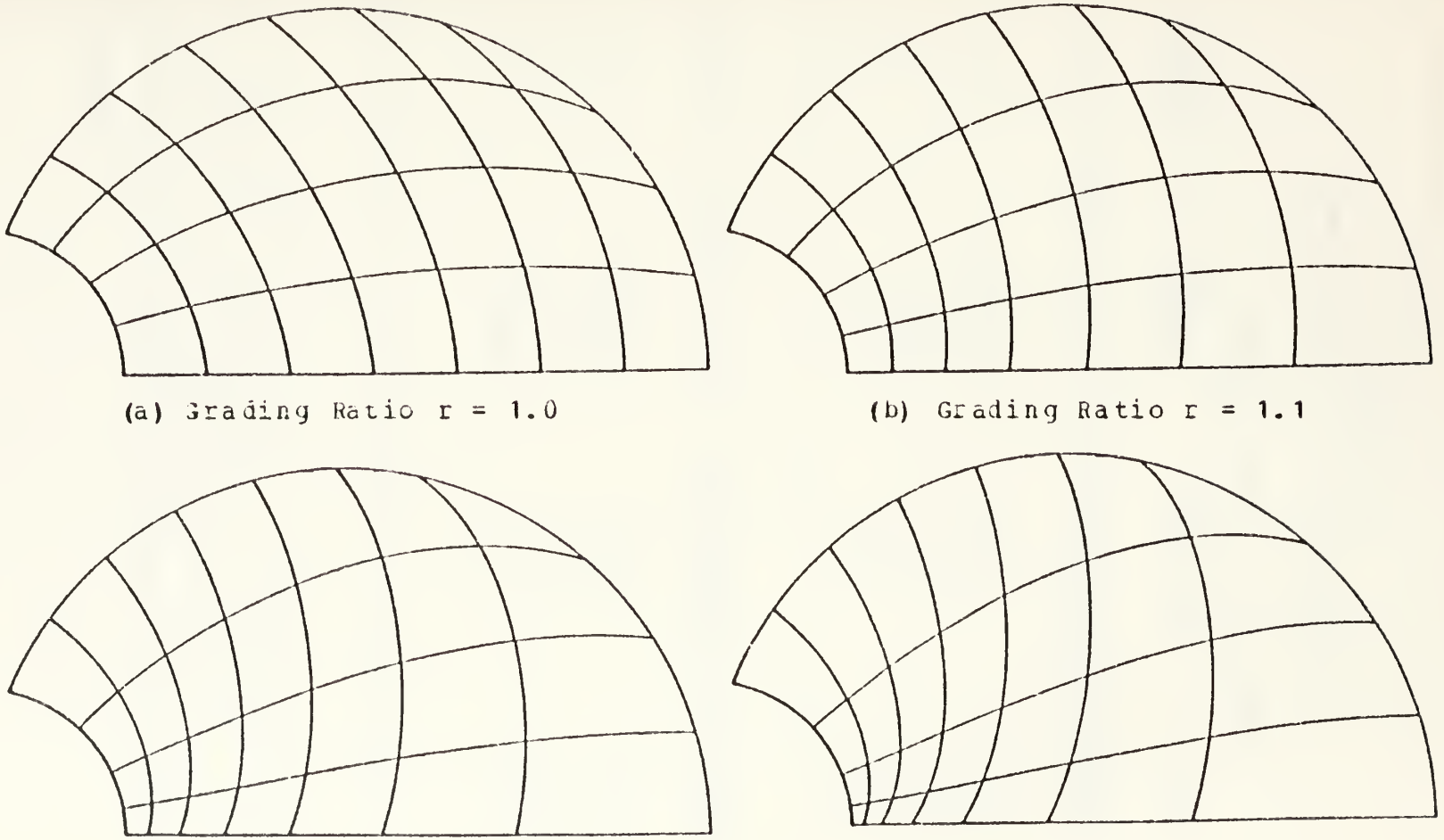
(a) Grading Ratio r = 1.0

(b) Grading Ratio r = 1.1

(c) Grading Ratio r = 1.2

(d) Grading Ratio r = 1.3

Figure 6.14 Mesh Grading - Linear Elements.



(a) Grading Ratio r = 1.0

(b) Grading Ratio r = 1.1

(c) Grading Ratio r = 1.2

(d) Grading Ratio r = 1.3

Figure 6.15 Mesh Grading - Quadratic Elements.

TABLE VIII

Mesh Grading Solution Results

Grading Ratio	ψ_{max}	$\tau_{max}/G\theta$	$I / G\theta$
	Percentage Relative Error		
Linear Element Grids (78 elements, 98 nodes, 72 degrees-of-freedom)			
1.0	0.060	-6.06	-1.77
1.1	0.161	-2.63	-1.56
1.2	0.389	-1.03	-1.77
1.3	0.679	-0.484	-2.20
Quadratic Element Grids (28 elements, 107 nodes, 76 degrees-of-freedom)			
1.0	-0.0093	-5.26	-0.0116
1.1	0.0063	-1.85	-0.0064
1.2	0.0162	-0.606	-0.0091
1.3	-0.0246	-0.413	-0.0179

It is likely that some of the error is attributable to the numerical inaccuracies due to element distortion. When applying a grading technique the analyst should seek an equitable balance between the refinement criterion and the grid topclogy.

Finally, since the grading ratio is usually applied to the nodal separation rather than the element edge lengths, it is advisable to reposition the midside nodes of quadratic elements so that they lie near the center of the element edges. This will generally improve the accuracy of all the solution parameters, especially if the grading is somewhat extreme.

E. GUIDELINES FOR TWO-DIMENSIONAL GRID OPTIMIZATION

In order to provide some guidelines for obtaining optimal finite element solutions for two-dimensional problems it is helpful to compare the solution results obtained for this problem employing the optimization techniques available. Such a comparison is presented in Table IX. The upper portion is for those techniques for which the initial analysis was performed using linear elements and the lower portion using quadratic elements. Note that all of the grids employ approximately the same number of degrees-of-freedom, which was the chosen measure of analysis cost in this investigation.

In making this comparison it is important to understand just how significant a change in error actually is. If the convergence rate of a solution parameter is very slow, even a small reduction in the error may require many more degrees-of-freedom. For this reason, the numbers in parentheses have been included by each error to provide a rough approximation of the number of degrees-of-freedom required to obtain similar accuracy using a uniform grid of the same number of degrees-of-freedom and elements of the same polynomial order as the initial analysis. In some cases, the analyses were not actually performed but instead the numbers in parentheses were obtained by extrapolation of the error versus degrees-of-freedom curve (Fig. 6.4), and ignoring round-off and ill-conditioning effects.

TABLE IX

Comparison of Two-Dimensional Results

Optimization Technique	No. DOF	No. Refine Cycles	Percentage Relative Error[1] ψ_{max}	τ_{max} / Gθ	T / Gθ
Linear Element Grids (Initial Analysis)					
Uniform	72	0	0.06	-6.06	-1.77
Contouring (SED)	77	1	0.28 (12)	-0.65 (1070)	-1.96 (65)
Selective (h-version)	75	2	-0.03 (140)	-0.90 (730)	-2.27 (55)
Selective (p-version)	76	1	-0.06 (68)	-2.67 (200)	-1.91 (67)
Subdomain Isolation	75	1	0.06 (72)	-0.61 (1150)	-1.59 (80)
Mesh Grading	72	-	0.39 (7)	-1.03 (620)	-1.77 (73)
Quadratic Element Grids (Initial Analysis)					
Uniform Quadratic	76	0	0.009	-5.26	-0.012
Selective (h-version)	74	3	-0.306 (20)	-0.14 (3470)	-0.039 (32)
Subdomain Isolation	70	1	0.009 (76)	-0.53 (850)	-0.009 (87)
Mesh Grading	76	-	0.016 (62)	-0.61 (740)	-0.009 (86)

[1]Parenthetic values indicate the approximate number of degrees-of-freedom required to achieve similar accuracy using a uniform grid of the same order as the initial analysis.

The first observation to be made from Table IX is that while all of the optimization techniques produced significant improvements in the accuracy of the maximum derivative quantity τ_{max}, the same cannot be said for the maximum solution quantity ψ_{max} and the integral quantity T. One might even conclude that grid optimization is not cost effective in the computation of these values since the uniform grid provides estimates which are nearly as accurate, and in some cases better, than the optimum grid. This conclusion would be correct if the solution maximum and its integral were the only resultants of interest in performing the analysis. Since the purpose of this study was to find an optimum grid which produced acceptable errors for all the resultants, the uniform grid is clearly inadequate. Moreover, since in the majority of engineering problems it is the derivative of the solution variable which is of primary interest, it deserves special consideration in making this comparison.

Furthermore, one might conclude that the reason the error in the maximum solution variable is larger for the optimal grid is because the strain energy density variation criterion always concentrates the degrees-of-freedom in the vicinity of the maximum derivative value, which in this case does not coincide with that of the maximum solution variable value. However, such a conclusion is incorrect and might erroneously lead one to attempt refinement where the maximum solution variable occurs in an effort to improve its estimate. Other investigations have revealed that in nearly all cases the maximum accuracies for all three solution resultants are obtained by refinement in the regions of highest strain energy density variation [Ref. 11]. The more likely source of increasing error for the optimal grids is the element distortion which was encountered in all but two techniques, selective p-version refinement and subdomain

isolation. Such distortion can be avoided but would require more sophisticated refinement techniques than were available for this investigation.

A reasonable choice for the optimum grid in Table IX would be one for which all three values in parentheses are as large as reasonably possible taking into consideration the number of cycles required to provide such accuracy. Based on such a criterion, the author is partial to subdomain isolation for the solution of two-dimensional problems using linear elements, and selective refinement for finite element solutions using quadratic elements. Clearly, before a concrete recommendation could be made for a wide range of applications, many more problems would have to be studied, but these two techniques were fairly simple to implement for a standard finite element code and the accelerated convergence of the solution resultants of interest was impressive. Conceivably, even greater solution accuracies might be obtained by using two or more of these techniques in combination.

Here again the crucial element in selecting the proper optimization strategy is the precise definition of the purpose for which the finite element analysis is to be performed. The results of Table IX tend to support the following recommendations and conclusions:

(1) Regardless of the optimization strategy chosen, higher order elements are indispensable if high accuracies for integral solution quantities are desired.

(2) If the maximum derivative of the solution variable is of greatest concern, the strictly local refinements employing subdomain isolation techniques can provide exceptional accuracy for a minimum number of refinement cycles.

(3) If the maximum solution variable value occurs at a point in the domain removed from the vicinity of the maximum derivative value, then its best estimate will likely be obtained using a reasonably fine uniform grid and selectively subdividing elements in the regions of large strain energy density variations.

VII. CONCLUSION AND RECOMMENDATIONS

The purpose of this paper has been to present an overview of some readily employed finite element grid optimization methods and to demonstrate their effectiveness in the application to some simple problems. This work is by no means all inclusive and the subject is still in its infancy. While there are many competing approaches to the problem, there is much more research to be done before any one becomes widely accepted as a standard analysis tool. Because of the limited time and resources available, some of the more sophisticated refinement criteria and techniques which have been developed have not been examined in any detail. Instead, the approach has been to examine those techniques which can be easily incorporated in a basic finite element code. However, it is likely that some recently developed and rather elaborate self-adaptive grid optimization codes will soon be available.

Also, this paper has not examined the important classes of problems in dynamic and nonlinear analysis. There is considerable ongoing research in the extension of these techniques to such problems, but the increased complexity is evident. For example, in vibration analysis there is an optimum grid for each unique eigenvalue, but it is for these types of problems that grid optimization is most promising.

At the beginning of this paper it was stated that the goal of grid optimization was to obtain maximum solution accuracy for a given analysis cost. Throughout this paper it has been shown that, prior to successfully embarking upon such a strategy, the underlying purpose of the analysis must be explicitly defined. Hopefully, it has been demonstrated that grid optimization is by no means an unrealistic goal

and is far more attractive than the non-adaptive practices widely used today.

The following are recommendations for future research topics:

(1) Investigation of more sophisticated refinement criteria based on element residuals and reliable error estimates.

(2) Investigation of grid optimization techniques employing adaptive application of the p-version.

(3) Implementation of a finite element preprocessor for performing hierarchical grid refinement.

(4) Implementation of a self-adaptive finite element code.

(5) Application of grid optimization techniques to problems of dynamic and nonlinear analysis.

APPENDIX A
FORMULATION OF THE ELASTIC CABLE PROBLEM

A. PROBLEM STATEMENT

Consider a perfectly elastic cable initially stretched between two fixed points a distance 2L apart and under tension T. If the cable bears a distributed load per unit

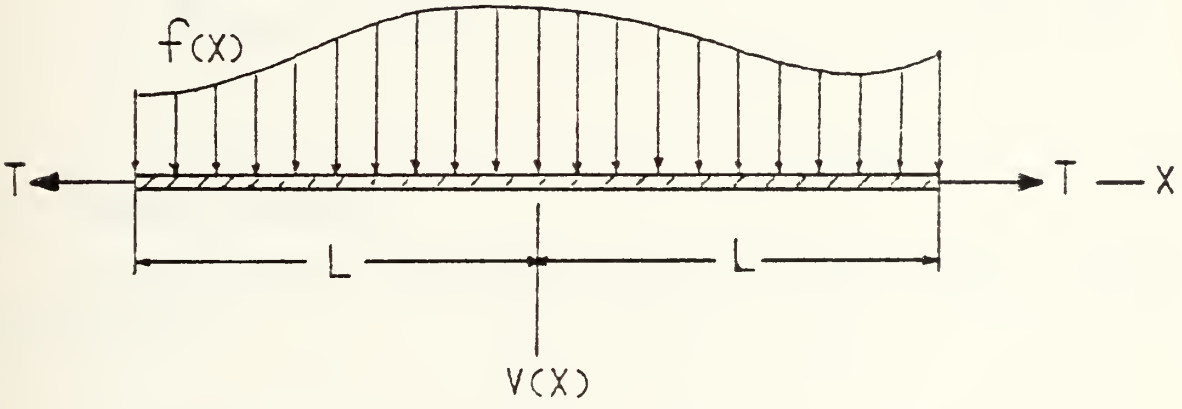


Figure A.1 Tension Cable Under Distributed Loading.

length f(x) as shown in Figure A.1, the governing differential equation for the downward defection v(x) is:

$$T \frac{\partial^2 v}{\partial x^2} + f(x) = 0 \qquad \text{(Eqn. A.1)}$$

subject to the essential boundary condition:

$$v\ (x = \pm L) = 0 \qquad \text{(Eqn. A.2)}$$

If the distributed load is a supportive load provided by a Winkler foundation of modulus k such that

$$f(x) = - k\ v(x)$$

and if a concentrated load P is applied at the midspan, Equation A.1 becomes:

$$T \frac{d^2 y}{dx^2} - k\ v = 0 \qquad \text{(Eqn. A.3)}$$

subject to the natural boundary condition:

$$\left.\frac{dv}{dx}\right|_{x=0^+} = - \frac{P/2}{T} \qquad \text{(Eqn. A.4)}$$

B. PROBLEM SOLUTION

The analytical solution of the two-point boundary value problem is:

$$v(x) = \frac{P/2}{T\lambda} \left[\tanh \lambda L \cosh \lambda x - \sinh \lambda x\right] \quad (0 \leq x \leq L) \qquad \text{(Eqn. A.5)}$$

where $\lambda = \sqrt{k/T}$.

The finite element solution is obtained by the Galerkin formulation using the consistent rather than the lumped approximation for the distributed loading.

APPENDIX B
FORMULATION OF THE TAPERED BAR PROBLEM

A. PROBLEM STATEMENT

Consider a tapered bar of length L and constant modulus of elasticity E fixed at one end. The cross-sectional area A(x) varies linearly from A_o at the fixed end to A_t at the tip. Let the bar be loaded axially by a concentrated tip load P, and a distributed load for which the intensity q(x) varies linearly from q_o at the fixed end to q_t at the tip as

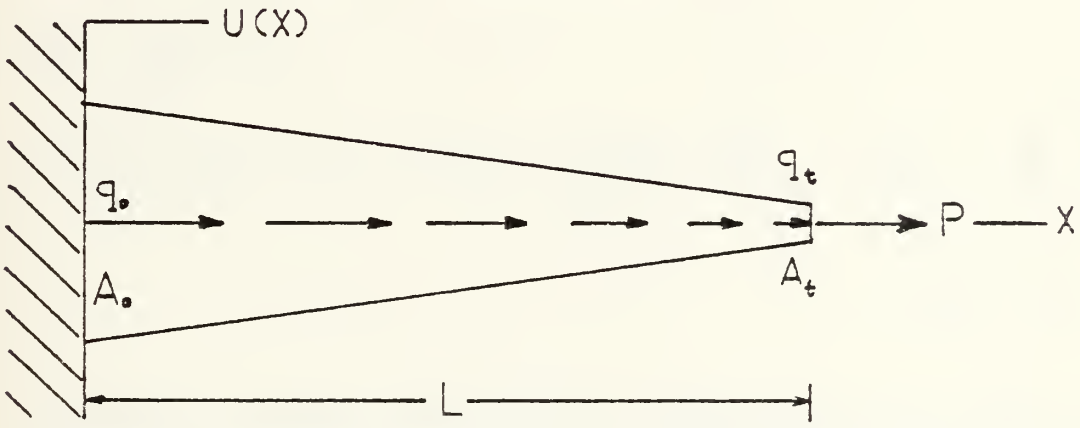


Figure B.1 Tapered Bar with Applied Loads.

shown in Figure B.1. The governing differential equation for the axial displacement u(x) is:

$$E\frac{d}{dx}\left[A\frac{du}{dx}\right] + q = 0 \qquad (0 \le x \le L) \qquad \text{(Eqn. B.1)}$$

subject to the essential boundary condition:

$$u\ (x = 0) = 0 \qquad \text{(Eqn. B.2)}$$

and the natural boundary condition:

$$\left.\frac{du}{dx}\right|_{x=L} = \frac{P}{EA_t} \qquad \text{(Eqn. B.3)}$$

B. PROBLEM SOLUTION

Let $\alpha = 1 - A_t/A_o$ and $\beta = 1 - q_t/q_o$.

For $F(x) = P + \int_x^L q(x)\,dx$ the solution is:

$$u(x) = \frac{1}{E}\int_0^x \frac{F(x)}{A(x)}\,dx$$

$$= \frac{q_o L}{\alpha A_o E}\left\{\left[\frac{\beta L}{2}\left(1-\frac{1}{\alpha^2}\right) - L\left(1-\frac{1}{\alpha}\right) - \frac{P}{q_o}\right] \ell n\left(1-\frac{\alpha x}{L}\right) + \left(1-\frac{\beta}{2\alpha}\right)x - \frac{\beta x^2}{4L}\right\} \qquad (0 \le x \le L)$$

(Eqn. B.4)

The finite element solution is obtained by the Galerkin formulation using the consistent rather than the lumped approximation for the distributed loading.

APPENDIX C
FORMULATION OF THE TORSION PROBLEM

A. PROBLEM STATEMENT

Consider a solid circular shaft of radius "a" and shear modulus G, with a circular groove, or keyway, of radius b along a generator of the shaft with the cross-section shown

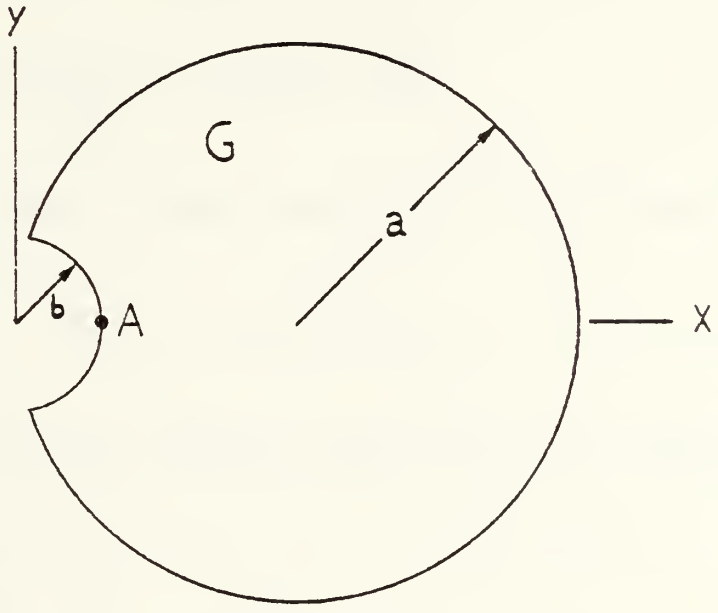


Figure C.1 Cross-section of Shaft with Keyway.

in Figure C.1. An applied torque T will produce an angle of twist per unit length θ. The equilibrium condition is satisfied if a torsional stress function ψ exists such that the shear stress components are:

$$\tau_{zx} = G\theta \frac{\partial \psi}{\partial y} \quad \text{and} \quad \tau_{zy} = -G\theta \frac{\partial \psi}{\partial x}$$

The governing differential equation for the torsional stress function $\psi(x,y)$ is:

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + 2 = 0 \qquad \text{(Eqn. C.1)}$$

subject to the Dirichlet condition, $\psi = 0$ on the boundary.

B. PROBLEM SOLUTION

The solution of Equation C.1 [Ref. 22: pp. 141-143] is:

$$\psi(x,y) = a\left(x - b^2\frac{x}{x^2+y^2}\right) - \tfrac{1}{2}(x^2+y^2) + \frac{b^2}{2} \qquad \text{(Eqn. C.2)}$$

The maximum shear stress occurs at point A (Fig. C.1) and is:

$$\tau_{max} = G\theta\,(2a - b) \qquad \text{(Eqn. C.3)}$$

The applied torque computed from the area integral:

$$T = 2G\theta \int_A \psi \, dA$$

$$= \frac{G\theta}{4}\left[(4a^4-8a^2b^2-2b^4)\alpha + (2a^3b+7ab^3)\sin\alpha\right] \qquad \text{(Eqn C.4)}$$

where $\alpha = \arccos(b/2a)$.
The strain energy per unit length of shaft is:

$$U' = \frac{1}{2G}\int_A \tau^2 \, dA = \tfrac{1}{2}T\theta \qquad \text{(Eqn. C.5)}$$

The variational formulation of the finite element approximation is presented in detail in Chapter 6 of Reference 25.

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